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FIXED POINT THEOREMS FOR TWO PAIRS OF MAPPINGS SATISFYING COMMON LIMIT RANGE PROPERTY IN *G* – METRIC SPACES

BY

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Abstract. The purpose of this paper is to prove a general fixed point theorem for two pairs of mappings in *G* - metric spaces, generalizing the results from (Popa and Patriciu, 2014) and unifying the results from (Giniswamy and Maheshwari, 2014). Also, a new result for a sequence of mappings is obtained. In the last part of this paper as applications, some fixed point results for mappings satisfying contractive conditions of integral type, for almost contractive mappings, for ϕ - contractive mappings and (ϕ,ψ) - contractive mappings in *G* - metric spaces, are obtained.

Keywords: fixed point; almost altering distance; common limit range property; implicit relation; *G* - metric space.

1. Introduction

Let (X,d) be a metric space and S,T be two mappings of X . In 1996, Jungck (Jungck, 1996) defined S and T to be compatible if

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$$\lim_{n\to\infty} d(TSx_n, STx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t,$$

for some $t \in X$.

This concept has been frequently used to prove the existence theorems in fixed point theory.

Let f, g be self mappings of a nonempty set X. A point $x \in X$ is a coincidence point of f and g if $w = fx = gx$ and w is said to be a point of coincidence of f and g. The set of all coincidence points of f and g is denoted by $C(f, g)$.

In 1994, Pant (Pant, 1994) introduced the notion of pointwise R - weakly commuting mapping, which is equivalent to commutativity at coincidence points.

In 1996, Jungck (Jungck, 1996) introduced the notion of weakly compatible mappings.

Definition 1.1 (Jungck, 1996) *Let X be a nonempty set and f, g be self mappings of X. f and g are weakly compatible if $fgu = gfu$ for all $u \in C(f, g)$.*

Hence, f and g are weakly compatible if and only if f and g are pointwise R - weakly commuting.

The study of common fixed points for noncompatible mappings is also interesting, the work of this regard beeing initiated by Pant in (Pant, 1998; 1999).

Aamri and El - Moutawakil (2002) introduced a generalization of noncompatible mappings.

Definition 1.2 (Aamri and El - Moutawakil, 2002) *Let S and T be two self mappings of a metric space (X, d). We say that S and T satisfy property (EA) if there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t,$$

for some $t \in X$.

Remark 1.1 *It is clear that two self mappings S and T of a metric space (X, d) will be noncompatible if there exists $\{x_n\}$ in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$ but $\lim_{n\to\infty} d(STx_n, TSx_n)$ is non zero or non existent.*

Therefore, two noncompatible self mappings of a metric space (X,d) satisfy property (EA).

It is known from (Pathak *et al.*, 2010) that the notions of weakly compatible mappings and mappings satisfying property (EA) are independent.

There exists a vast literature concerning the study of fixed points for pairs of mappings satisfying property (EA).

In 2005, Liu *et al*. (Liu *et al.*, 2005) defined the notion of common property (EA).

Definition 1.3 (Liu *et al.*, 2005) *Two pairs (A,S) and (B,T) of self mappings of a metric space (X,d) are said to satisfy common property (EA) if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t,$$

for some $t \in X$.

In 2011, Sintunavarat and Kumam (Sintunavarat and Kumam, 2011) introduced the notion of common limit range property.

Definition 1.4 (Sintunavarat and Kumam, 2011) *A pair (A,S) of self mappings of a metric space (X,d) is said to satisfy the common limit range property with respect to S, denoted $CLR_{(S)}$ if there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t,$$

for some $t \in S(X)$.

Thus we can infer that a pair (A,S) satisfying the property (EA) along with the closedness of the subspace $S(X)$ always has the $CLR_{(S)}$ - property with respect to S (see Examples 2.16, 2.17 (Imdad *et al.*, 2012)).

Recently, Imdad *et al.* (2013) extended the notion of common limit range property to the pairs of self mappings.

Definition 1.5 (Imdad *et al.*, 2013) *Two pairs (A,S) and (B,T) of self mappings of a metric space (X,d) are said to satisfy common limit range property with respect to S and T, denoted $CLR_{(S,T)}$ if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t,$$

where $t \in S(X) \cap T(X)$.

Some fixed point results for pairs of mappings with $CLR_{(S,T)}$ property are obtained in (Imdad and Chauhan, 2013; Karapinar *et al.*, 2013) and in other papers.

2. Preliminaries

In (Dhage, 1992; 2000), Dhage introduced a new class of generalized metric space, named D - metric spaces. Mustafa and Sims (2003; 2006), proved that most of the claims concerning the fundamental topological structures on D - metric spaces are incorrect and introduced appropriate notion of generalized metric space, named G - metric space. In fact, Mustafa, Sims and other authors studied many fixed point results for self mappings under certain conditions in (Mustafa *et al.*, 2008; Mustafa and Sims, 2009; Shatanawi, 2010), and in other papers.

Definition 2.1 (Mustafa and Sims, 2006) *Let X be a nonempty set and $G: X^3 \to \mathbb{R}_+$ be a function satisfying the following properties:*

$(G_1): G(x,y,z)=0$ *for* $x=y=z$,

$(G_2): 0<G(x,x,y)$ *for all* $x,y\in X$ *with* $x\neq y$,

$(G_3): G(x,y,y)\leq G(x,y,z)$ *for all* $x,y,z\in X$ *with* $z\neq y$,

$(G_4): G(x,y,z)=G(y,z,x)=G(z,x,y)=\ldots$ (*symmetry in all three variables*),

$(G_5): G(x,y,z)\leq G(x,a,a)+G(a,y,z)$ *for all* $x,y,z,a\in X$ (*triangle inequality*).

The function G is called a G - metric on X and (X,G) is called a G - metric space.

Note that if $G(x,y,z)=0$, then $x=y=z$.

Remark 2.1 *Let (X,G) be a G - metric space. If $y=z$, then $G(x,y,y)$ is a quasi - metric on X. Hence, (X,Q), where $Q(x,y)=G(x,y,y)$, is a quasi - metric space and since every metric space is a particular case of quasi - metric space it follows that the notion of G - metric space is a generalization of a metric space.*

Definition 2.2 (Mustafa and Sims, 2006) *Let (X,G) be a G - metric space. A sequence $\{x_n\}$ in X is said to be:*

a) G - convergent if for $\varepsilon>0$, there exist $x\in X$ and $k\in\mathbb{N}$ such that for all $m,n\in\mathbb{N}, m,n\geq k$, $G(x,x_n,x_m)<\varepsilon$.

b) G - Cauchy if for $\varepsilon>0$, there exists $k\in\mathbb{N}$ such that for all $m,n,p\in\mathbb{N}$, $m,n,p\geq k$, $G(x_n,x_m,x_p)<\varepsilon$, that is $G(x_n,x_m,x_p)\to 0$ as $n,m,p\to\infty$.

c) A G - metric space is said to be G - complete if every G - Cauchy sequence in X is G - convergent.

Lemma 2.1 (Mustafa and Sims, 2006) *Let (X,G) be a G - metric space. Then, the following conditions are equivalent:*

1) *$\{x_n\}$ is G - convergent to x;*

2) *$G(x_n,x_n,x)\to 0$ as $n\to\infty$;*

3) $G(x_n, x, x) \to 0$ *as* $n \to \infty$*;*

4) $G(x_n, x_m, x) \to 0$ *as* $n, m \to \infty$.

Lemma 2.2 (Mustafa and Sims, 2006) *If* (X, G) *is a* G *- metric space, then the following conditions are equivalent:*

1) $\{x_n\}$ *is* G *- Cauchy;*

2) *For* $\varepsilon > 0$*, there exists* $k \in \mathbb{N}$ *such that* $G(x_n, x_m, x_m) < \varepsilon$ *for all* $m, n \in \mathbb{N}$, $m, n \geq k$.

Lemma 2.3 (Mustafa and Sims, 2006) *Let* (X, G) *be a* G *- metric space. Then, the function* $G(x, y, z)$ *is jointly continuous in all three of its variables.*

Definition 2.3 (Mustafa and Sims, 2006) *A* G *- metric on a set* X *is said to be symmetric if* $G(x, y, y) = G(y, x, x)$ *for all* $x, y \in X$. *Then,* (X, G) *is said to be symmetric* G *- metric space.*

Quite recently (Popa and Patriciu, 2014), a general fixed point theorem for a pair of mappings satisfying $CLR_{(S)}$ - property in G - metric spaces is proved.

Definition 2.4 (Khan *et al.*, 1984) *An altering distance is a function* $\phi : [0, \infty) \to 0, \infty)$ *satisfying:*

(ϕ_1): ϕ *is increasing and continuous;*

(ϕ_2): $\phi(t) = 0$ *if and only if* $t = 0$.

Fixed point theorems involving altering distances have been studied in (Popa and Mocanu, 2007; Sastri and Babu, 1998; 1999) and in other papers.

Definition 2.5 (Popa and Patriciu, 2014) *A function* $\psi : [0, \infty) \to 0, \infty)$ *is an almost altering distance if:*

(ψ_1): ψ *is continuous;*

(ψ_2): $\psi(t) = 0$ *if and only if* $t = 0$.

Remark 2.1 *Every altering distance is an almost altering distance, but the converse is not true.*

Example 2.1 $\psi(t) = \begin{cases} t, t \in [0,1] \\ \dfrac{1}{t}, t \in (1, \infty). \end{cases}$

3. Implicit Relations in G - Metric Spaces

Several fixed point theorems and common fixed point theorems have been unified considering a general condition by an implicit function in (Popa, 1997; 1999) and in other papers.

Recently, the method is used in the study of fixed points in metric spaces, symmetric spaces, quasi - metric spaces, b - metric spaces, ultra - metric spaces, reflexive spaces, compact metric spaces, paracompact metric spaces, in two and three metric spaces, for single - valued mappings, hybrid pairs of mappings and set - valued mappings. The method is used in the study of fixed points for mappings satisfying a contractive/extensive condition of integral type, in fuzzy metric spaces, probabilistic metric spaces, intuitionistic metric spaces, partial metric spaces and G - metric spaces.

The study of fixed points for mappings satisfying implicit relations in G - metric spaces is initiated in (Popa and Patriciu, 2012; 2013) and in other papers.

With this method the proofs of some fixed point theorems are more simple. Also, the method allows the study of local and global properties of fixed point structures.

The study of fixed points for pairs of self mappings with common limit range property in metric spaces satisfying implicit relations is initiated in (Imdad and Chauhan, 2013).

The study of fixed points for a pair of self mappings with common limit range property in G - metric spaces is initiated in (Popa and Patriciu, 2014).

In 2008, Ali and Imdad (Ali and Imdad, 2008) introduced a new class of implicit relations.

Definition 3.1 (Ali and Imdad, 2008) *Let* $\mathcal{F}_G$ *be the family of lower semi - continuous functions* $F:\mathbb{R}_+^6\to\mathbb{R}$ *satisfying the following conditions:*

(F_1): $F(t,0,t,0,0,t)>0$, *for all* $t>0$;

(F_2): $F(t,0,0,t,t,0)>0$, *for all* $t>0$;

(F_3): $F(t,t,0,0,t,t)>0$, *for all* $t>0$.

Example 3.1 $F(t_1,...,t_6)=t_1-at_2-bt_3-ct_4-dt_5-et_6$, *where* $a,b,c,d,e\geq 0$ *and* $a+b+c+d+e<1$.

Example 3.2 $F(t_1,...,t_6)=t_1-k\max\left\{t_2,t_3,t_4,\dfrac{t_5+t_6}{2}\right\}$, *where* $k\in[0,1)$.

Example 3.3 $F(t_1,...,t_6)=t_1-k\max\{t_2,t_3,...,t_6\}$, *where* $k\in[0,1)$.

Example 3.4 $F(t_1,...,t_6)=t_1-k\max\left\{t_2,\dfrac{t_3+t_4}{2},\dfrac{t_5+t_6}{2}\right\}$, *where* $k\in[0,1)$.

Example 3.5 $F(t_1,...,t_6)=t_1-at_2-b\max\{t_3,t_4\}-c\max\{t_2,t_5,t_6\}$, *where* $a,b,c\geq 0$ *and* $a+b+c<1$.

Example 3.6 $F(t_1,...,t_6)=t_1-\alpha\max\{t_2,t_3,t_4\}-(1-\alpha)(at_5+bt_6)$, *where* $\alpha\in(0,1)$, $a,b\geq 0$ *and* $a+b<1$.

Example 3.7 $F(t_1,...,t_6) = t_1 - at_2 - b(t_3 + t_4) - c\min\{t_5, t_6\}$, *where* $a,b,c > 0$ *and* $a + b + c < 1$.

Example 3.8 $F(t_1,...,t_6) = t_1 - at_2 - \dfrac{b(t_5 + t_6)}{1 + t_3 + t_4}$, *where* $a,b \geq 0$ *and* $a + 2b < 1$.

Example 3.9 $F(t_1,...,t_6) = t_1 - \max\{ct_2, ct_3, ct_4, at_5 + bt_6\}$, *where* $c \in (0,1)$, $a,b \geq 0$ *and* $a + b + c < 1$.

Quite recently, the following theorem is proved in (Popa & Patriciu, 2014).

Theorem 3.1 (Popa & Patriciu, 2014) *Let* T *and* S *be self mappings of a* G *- metric space* (X,G) *such that*

$$F(\psi(G(Tx,Tx,Ty)),\psi(G(Sx,Sx,Sy)),\psi(G(Tx,Tx,Sx)),$$
$$\psi(G(Ty,Ty,Sy)),\psi(G(Sx,Sx,Ty)),\psi(G(Tx,Sx,Sy))) < 0,$$

for all $x, y \in X$, *where* F *satisfies properties* $(F_1),(F_3)$ *and* ψ *is an almost altering distance. If* T *and* S *satisfy* $CLR_{(S)}$ *- property, then* $C(T,S) \neq \varnothing$. *Moreover, if* T *and* S *are weakly compatible, then* T *and* S *have a unique common fixed point.*

The purpose of this paper is to prove a general fixed point theorem for two pairs of mappings satisfying common limit range property in G - metric spaces, generalizing the results from (Popa and Patriciu, 2014) and unifying the results from (Giniswamy and Maheshwari, 2014). Also, a new result for a sequence of mappings is obtained.

In the last part of this paper, as applications, some fixed point results for mappings satisfying contractive conditions of integral type, for almost contractive mappings, for φ - contractive mappings and (φ,ψ) - contractive mappings in G - metric spaces are obtained.

4. Main Results

Lemma 4.1 (Abbas and Rhoades, 2009) *Let* f, g *be two weakly compatible self mappings of a nonempty set* X. *If* f *and* g *have a unique point of coincidence* $w = fx = gx$ *for some* $x \in X$, *then* w *is the unique common fixed point of* f *and* g.

Theorem 4.1 *Let* A, B, S *and* T *be self mappings of a* G *- metric space* (X,G) *satisfying inequality*

$$F(\psi(G(Ax,By,By)),\psi(G(Sx,Ty,Ty)),\psi(G(Sx,Sx,Ax)),$$
$$\psi(G(Ty,By,By)),\psi(G(Sx,By,By)),\psi(G(Ax,Ty,Ty))) \leq 0, \quad (4.1)$$

for all $x, y \in X$ *,* F *satisfies property* (F_3) *and* ψ *is an almost altering distance.*

If there exist $u, v \in X$ *such that* $Au = Su$ *and* $Bv = Tv$ *, then there exists* $t \in X$ *such that* t *is the unique point of coincidence of* A *and* S*, as well* t *is the unique point of coincidence of* B *and* T *.*

Proof. First we prove that $Su = Tv$. Suppose that $Su \neq Tv$. By (4.1) we obtain

$$F(\psi(G(Au, Bv, Bv)), \psi(G(Su, Tv, Tv)), \psi(G(Su, Su, Au)),$$
$$\psi(G(Tv, Bv, Bv)), \psi(G(Su, Bv, Bv)), \psi(G(Au, Tv, Tv))) \leq 0,$$

$$F(\psi(G(Su, Tv, Tv)), \psi(G(Su, Tv, Tv)), 0, 0, \psi(G(Su, Tv, Tv)), \psi(G(Su, Tv, Tv))) \leq 0,$$

a contradiction of (F_3).

Hence, $Su = Tv$, which implies $Su = Au = Bv = Tv = t$. Suppose that there exists $z = Aw = Sw$ with $z \neq t$. Then, by (4.1) we obtain

$$F(\psi(G(Aw, Bv, Bv)), \psi(G(Sw, Tv, Tv)), \psi(G(Sw, Sw, Aw)),$$
$$\psi(G(Tv, Bv, Bv)), \psi(G(Sw, Bv, Bv)), \psi(G(Aw, Tv, Tv))) \leq 0,$$

$$F(\psi(G(Sw, Tv, Tv)), \psi(G(Sw, Tv, Tv)), 0, 0, \psi(G(Sw, Tv, Tv)), \psi(G(Sw, Tv, Tv))) \leq 0,$$

a contradiction of (F_3).

Hence, $z = Sw = Aw = Tv = Bv = Au = Su = t$ and t is the unique point of coincidence of A and S. Similarly, t is the unique point of coincidence of B and T.

Theorem 4.2 *Let* A, B, S *and* T *be self mappings of a* G *- metric space* (X, G) *satisfying inequality (4.1) for all* $x, y \in X$ *,* $F \in \mathcal{F}_G$ *and* ψ *is an almost altering distance. If* (A, S) *and* (B, T) *satisfy* $CLR_{(S,T)}$ *- property, then*

i) $C(A, S) \neq \emptyset$*,*

ii) $C(B, T) \neq \emptyset$*.*

Moreover, if (A, S) *and* (B, T) *are weakly compatible, then* A, B, S *and* T *have a unique common fixed point.*

Proof. Since (A, S) and (B, T) satisfy $CLR_{(S,T)}$ - property, there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z,$$

where $z \in S(X) \cap T(X)$.

Since $z \in T(X)$, there exists $u \in X$ such that $z = Tu$.

By (4.1) we have

$$F(\psi(G(Ax_n, Bu, Bu)), \psi(G(Sx_n, Tu, Tu)), \psi(G(Sx_n, Sx_n, Ax_n)),$$
$$\psi(G(Tu, Bu, Bu)), \psi(G(Sx_n, Bu, Bu)), \psi(G(Ax_n, Tu, Tu))) \leq 0.$$

Letting n tends to infinity we obtain

$$F(\psi(G(z,Bu,Bu)),0,0,\psi(G(z,Bu,Bu)),\psi(G(z,Bu,Bu)),0)\le 0,$$

a contradiction of (F_2) if $\psi(G(z,Bu,Bu))>0$. Hence, $\psi(G(z,Bu,Bu))=0$, which implies $z=Bu=Tu$ and $C(B,T)\neq\varnothing$.

Since $z\in S(X)$, there exists $v\in X$ such that $z=Sv$. By (4.1) we obtain

$$F(\psi(G(Av,Bu,Bu)),\psi(G(Sv,Tu,Tu)),\psi(G(Sv,Sv,Av)),$$
$$\psi(G(Tu,Bu,Bu)),\psi(G(Sv,Bu,Bu)),\psi(G(Av,Tu,Tu)))\le 0,$$

$$F(\psi(G(Av,z,z)),0,\psi(G(Av,z,z)),0,0,\psi(G(Av,z,z)))\le 0,$$

a contradiction of (F_1) if $\psi(G(Av,z,z))>0$. Hence, $\psi(G(Av,z,z))=0$, which implies $z=Av=Sv$ and $C(A,S)\neq\varnothing$.

By Theorem 4.1, z is the unique point of coincidence of (A,S) and (B,T).

Moreover, if (A,S) and (B,T) are weakly compatible, by Lemma 4.1, z is the unique fixed point of A,B,S and T.

If $\psi(t)=t$, then by Theorem 4.2 we obtain

Theorem 4.3 *Let A,B,S and T be self mappings of a G - metric space (X,G) satisfying the inequality*

$$F(G(Ax,By,By),G(Sx,Ty,Ty),G(Sx,Sx,Ax),$$
$$G(Ty,By,By),G(Sx,By,By),G(Ax,Ty,Ty))\le 0, \tag{4.2}$$

for all $x,y\in X$, $F\in \mathcal{F}_G$.

If (A,S) and (B,T) satisfy $CLR_{(S,T)}$ - property, then

i) $C(A,S)\neq\varnothing$,

ii) $C(B,T)\neq\varnothing$.

Moreover, if (A,S) and (B,T) are weakly compatible, then A,B,S and T have a unique common fixed point.

Example 4.1 Let $X=[0,11]$ and let $G:X^3\to\mathbb{R}_+$ be the $G-$ metric defined as follows

$$G(x,y,z)=\max\{|x-y|,|y-z|,|x-z|\}$$

for all $x,y,z\in X$. Then (X,G) is a $G-$ metric space.

Define the self mappings A,B,S and T

$$Ax = \begin{cases} 2, x \in [0,2] \cup (5,11] \\ 5, x \in (2,5], \end{cases} \qquad Sx = \begin{cases} 2, x \in [0,2] \\ 6, x \in (2,5) \\ \dfrac{3x+1}{8}, x \in [5,11], \end{cases}$$

$$Bx = \begin{cases} 2, x \in [0,2] \cup (5,11] \\ 4, x \in (2,5], \end{cases} \qquad Tx = \begin{cases} 2, x \in [0,2] \\ 8, x \in (2,5] \\ x-3, x \in (5,11]. \end{cases}$$

Then

$$AX = \{2,5\}, BX = \{2,4\}, SX = \left[2, \frac{17}{4}\right) \cup \{6\}, TX = [2,8].$$

Let $x_n = 2 - \frac{1}{n}$ and $y_n = 2 - \frac{1}{n^2}$ be. Then

$$\lim Ax_n = \lim Sx_n = \lim By_n = \lim Ty_n = 2 \in S(X) \cap T(X)$$

and (A,S) and (B,T) satisfies $CLR_{(S,T)}$ – property.

On the other hand, $z = 2$ is the unique point of coincidence of (A,S) and (B,T).

$Ax = Sx$ for $x \in [0,2]$, $Bx = Tx$ for $x \in [0,2]$, $ASx = SAx = 2$. Similarly, $BTx = TBx = 2$, hence (A,S) and (B,T) are weakly compatible.

If

$$M(x,y) = \max\{G(Sx,Ty,Ty), G(Sx,Sx,Ax), G(Ty,By,By), G(Sx,By,By), G(Ax,Ty,Ty)\},$$

then by a routine calculation we obtain

$$G(Ax,By,By) \le kM(x,y),$$

with $k \in \left[\frac{3}{4}, 1\right)$.

Thus, by Example 1 and Theorem 4.2, A, B, S and T have a unique common fixed point which is $x = 2$.

Similarly as in Theorem 4.2 we obtain

Theorem 4.4 *Let A,B,S and T be self mappings of a G - metric space (X,G) satisfying inequality*

$$F(\psi(G(Ax,Ax,By)), \psi(G(Sx,Sx,Ty)), \psi(G(Sx,Ax,Ax)), \psi(G(Ty,Ty,By)), \psi(G(Sx,Sx,By)), \psi(G(Ax,Ax,Ty))) \le 0, \tag{4.3}$$

for all $x, y \in X$, $F \in \mathcal{F}_G$ and ψ is an almost altering distance.

If (A,S) *and* (B,T) *satisfy* $CLR_{(S,T)}$ *- property, then*

i) $C(A,S)\neq\emptyset$,

ii) $C(B,T)\neq\emptyset$.

Moreover, if (A,S) *and* (B,T) *are weakly compatible, then* A,B,S *and* T *have a unique common fixed point.*

Theorem 4.5 *Let* (X,G) *be a* G *- metric space and* A,B,S *and* T *be self mappings of* X *satisfying the inequality*

$$\begin{aligned}G(Ax,By,By)&\leq aG(Sx,Ty,Ty)+bG(Sx,Sx,Ax)+\\&+cG(Ty,By,By)+dG(Sx,By,By)+eG(Ax,Ty,Ty),\end{aligned}\tag{4.4}$$

for all $x,y\in X$, $a,b,c,d,e\geq 0$ *and* $a+b+c+d+e<1$.

If (A,S) *and* (B,T) *satisfy* $CLR_{(S,T)}$ *- property, then*

i) $C(A,S)\neq\emptyset$,

ii) $C(B,T)\neq\emptyset$.

Moreover, if (A,S) *and* (B,T) *are weakly compatible, then* A,B,S *and* T *have a unique common fixed point.*

Corollary 4.1 (**Theorem 2.5** (Giniswamy and Maheshwari, 2014)) *Let* (X,G) *be a* G *- metric space and* A,B,S *and* T *be self mappings of* X *such that:*

1) (A,S) and (B,T) satisfy $CLR_{(S,T)}$ - property;

2)
$$\begin{aligned}G(Ax,By,Bz)&\leq pG(Sx,Ty,Ty)+qG(Sx,Sx,Ax)+\\&+rG(Ty,Bz,Bz)+t[G(Ax,Ty,Tz)+G(Sx,By,Bz)],\end{aligned}\tag{4.5}$$

for all $x,y,z\in X$ *, where* $p,q,r,t\geq 0$ *and* $p+q+r+2t<1$.

Then (A,S) *and* (B,T) *have a unique point of coincidence in* X .

Moreover, if (A,S) *and* (B,T) *are weakly compatible, then* A,B,S *and* T *have a unique common fixed point.*

Proof. Let $y=z$, then by (4.5) we obtain a particular case of (4.4) and the proof follows from Theorem 4.5.

Theorem 4.6 *Let* (X,G) *be a* G *- metric space and* A,B,S *and* T *be self mappings of* X *satisfying the inequality:*

$$\begin{aligned}G(Ax,By,By)\leq k\max\{&G(Sx,Ty,Ty),G(Sx,Sx,Ax),\\&G(Ty,By,By),\frac{G(Sx,By,By)+G(Ax,Ty,Ty)}{2}\},\end{aligned}\tag{4.6}$$

for all $x,y\in X$ *and* $k\in[0,1)$.

If (A,S) *and* (B,T) *satisfy* $CLR_{(S,T)}$ *- property, then*

i) $C(A,S)\neq\emptyset$,

ii) $C(B,T)\neq\emptyset$.

Moreover, if (A,S) *and* (B,T) *are weakly compatible, then* A,B,S *and* T *have a unique common fixed point.*

Proof. The proof follows from Theorem 4.3 and Example 3.2.

Corollary 4.2 (**Theorem 2.6** (Giniswamy and Maheshwari, 2014)) *Let* (X,G) *be a* G *- metric space and* A,B,S *and* T *be self mappings of* X *such that:*

1) (A,S) *and* (B,T) *satisfy* $CLR_{(S,T)}$ *- property;*

2) $G(Ax,By,Bz) \le hu(x,y,z)$, *where* $h \in (0,1)$, $x,y,z \in X$ *and*

$$u(x,y,z) \in \{G(Ax,Sx,Sx), G(Sx,Ty,Ty), G(Ty,By,By), \frac{G(Ax,Ty,Tz)+G(Sx,By,Bz)}{2}\}.$$

Then (A,S) *and* (B,T) *have a unique point of coincidence in* X.

Moreover, if (A,S) *and* (B,T) *are weakly compatible, then* A,B,S *and* T *have a unique common fixed point.*

Proof. Let $y = z$, then by (2) we obtain

$$G(Ax,By,By) \le h\max\{G(Sx,Ty,Ty), G(Sx,Sx,Ax),$$
$$G(Ty,By,By), \frac{G(Sx,By,By)+G(Ax,Ty,Ty)}{2}\},$$

which is inequality (4.6) and the proof of Corollary 4.2 follows from Theorem 4.6.

For a function $f : X \to X$ we denote

$$Fix(f) = \{x \in X : x = fx\}.$$

Theorem 4.7 *Let* A,B,S *and* T *be self mappings of a* G *- metric space* (X,G). *If the inequality (4.1) holds for all* $x,y \in X$, $F \in \mathcal{F}_G$ *and* ψ *is an almost altering distance, then*

$$[Fix(S) \cap Fix(T)] \cap Fix(A) = [Fix(S) \cap Fix(T)] \cap Fix(B).$$

Proof. Let $x \in [Fix(S) \cap Fix(T)] \cap Fix(A)$. Then by (4.1) we have

$$F(\psi(G(Ax,Bx,Bx)), \psi(G(Sx,Tx,Tx)), \psi(G(Sx,Sx,Ax)),$$
$$\psi(G(Tx,Bx,Bx)), \psi(G(Sx,Bx,Bx)), \psi(G(Ax,Tx,Tx))) \le 0,$$
$$F(\psi(G(x,Bx,Bx)),0,0,\psi(G(x,Bx,Bx)),\psi(G(x,Bx,Bx)),0) \le 0,$$

a contradiction of (F_2) if $\psi(G(x,Bx,Bx)) > 0$. Hence, $\psi(G(x,Bx,Bx)) = 0$ which implies $x = Bx$ and $x \in Fix(B)$.

Hence

$$[Fix(S) \cap Fix(T)] \cap Fix(A) \subset [Fix(S) \cap Fix(T)] \cap Fix(B).$$

Similarly, by (4.1) and (F_1) we obtain

$$[Fix(S) \cap Fix(T)] \cap Fix(B) \subset [Fix(S) \cap Fix(T)] \cap Fix(A).$$

Theorems 4.2 and 4.7 imply the following one.

Theorem 4.8 *Let* S,T *and* $\{A_i\}_{i\in\mathbb{N}^*}$ *be self mappings of a* G *- metric space* (X,G) *satisfying the inequality*

$$\begin{aligned}&F(\psi(G(A_i x, A_{i+1}y, A_{i+1}y)), \psi(G(Sx,Ty,Ty)), \psi(G(Sx,Sx,A_i x)),\\ &\psi(G(Ty, A_{i+1}y, A_{i+1}y)), \psi(G(Sx, A_{i+1}y, A_{i+1}y)), \psi(G(A_i x, Ty, Ty))) \le 0,\end{aligned} \tag{4.7}$$

for all $x,y\in X$, $F\in\mathcal{F}_G$, ψ *is an almost altering distance and* $i\in\mathbb{N}^*$.

If (A_1,S) *and* (A_2,T) *satisfy* $CLR_{(S,T)}$ *- property and* $(A_1,S),(A_2,T)$ *are weakly compatible, then* S,T *and* $\{A_i\}_{i\in\mathbb{N}^*}$ *have a unique common fixed point.*

If $\psi(t)=t$, from Theorem 4.8 we obtain

Theorem 4.9 *Let* S,T *and* $\{A_i\}_{i\in\mathbb{N}^*}$ *be self mappings of a* G *- metric space* (X,G) *satisfying the inequality*

$$\begin{aligned}&F(G(A_i x, A_{i+1}y, A_{i+1}y), G(Sx,Ty,Ty), G(Sx,Sx,A_i x),\\ &G(Ty, A_{i+1}y, A_{i+1}y), G(Sx, A_{i+1}y, A_{i+1}y), G(A_i x, Ty, Ty)) \le 0,\end{aligned} \tag{4.8}$$

for all $x,y\in X$, $F\in\mathcal{F}_G$ *and* $i\in\mathbb{N}^*$.

If (A_1,S) *and* (A_2,T) *satisfy* $CLR_{(S,T)}$ *- property and* $(A_1,S),(A_2,T)$ *are weakly compatible, then* S,T *and* $\{A_i\}_{i\in\mathbb{N}^*}$ *have a unique common fixed point.*

Remark 4.1 *We obtain similar results from Theorem 4.4.*

5. Applications

5.1. Fixed Points for Mappings Satisfying Contractive Conditions of Integral Type

In (Branciari, 2002), Branciari established the following theorem which opened the way to the study of fixed points for mappings satisfying contractive conditions of integral type.

Theorem 5.1 (Branciari, 2002) *Let* (X,d) *be a complete metric space,* $c\in(0,1)$ *and* $f:X\to X$ *such that for all* $x,y\in X$

$$\int_0^{d(fx,fy)} h(t)dt \le c\int_0^{d(x,y)} h(t)dt\,,$$

whenever $h:[0,\infty)\to[0,\infty)$ *is a Lebesgue measurable mapping which is summable (i.e., with finite integral) on each compact subset of* $[0,\infty)$ *such that*

$\int_0^{\varepsilon} h(t)dt > 0$ *for each* $\varepsilon > 0$. *Then, f has an unique fixed point* $z \in X$ *such that for all* $x \in X$, $z = \lim_{n\to\infty} f^n x$.

Theorem 5.1 has been extended to a pair of compatible mappings in (Kumar *et al.*, 2007).

Theorem 5.2 (Kumar *et al.*, 2007) *Let* f, g *be compatible mappings of a complete metric space with g – continuous satisfying the following conditions:*

1) $f(X) \subset g(X)$,

2) $\int_0^{d(fx,gy)} h(t)dt \le c \int_0^{d(x,y)} h(t)dt$,

for some $c \in (0,1)$, *whenever* $x, y \in X$ *and* $h(t)$ *as in Theorem 5.1.*

Then, f *and* g *have a unique common fixed point.*

Some fixed point results for mappings satisfying contractive conditions of integral type are proved in (Popa and Mocanu, 2007; 2009) and in other papers.

Lemma 5.1 Let $h:[0,\infty) \to 0,\infty)$ *as in Theorem 5.1. Then* $\psi(t) = \int_0^t h(x)dx$ *is an almost altering distance.*

Proof. The proof follows from Lemma 2.5 (Popa and Mocanu, 2009).

Theorem 5.3 *Let* A, B, S *and* T *be self mappings of a* G *- metric space* (X, G) *such that*

$$\begin{aligned} &F(\int_0^{G(Ax,By,By)} h(t)dt, \int_0^{G(Sx,Ty,Ty)} h(t)dt, \int_0^{G(Sx,Sx,Ax)} h(t)dt,\\ &\int_0^{G(Ty,By,By)} h(t)dt, \int_0^{G(Sx,By,By)} h(t)dt, \int_0^{G(Ax,Ty,Ty)} h(t)dt) \le 0, \end{aligned} \tag{5.1}$$

for all $x, y \in X$, *where* $F \in \mathcal{F}_G$ *and* $h(t)$ *as in Theorem 5.1.*

If (A,S) *and* (B,T) *satisfy* $CLR_{(S,T)}$ *- property, then*

i) $C(A,S) \ne \emptyset$,

ii) $C(B,T) \ne \emptyset$.

Moreover, if (A,S) *and* (B,T) *are weakly compatible, then* A, B, S *and* T *have a unique common fixed point.*

Proof. By Lemma 5.1, $\psi(t) = \int_0^t h(x)dx$ is an almost altering distance. By (5.1) we have

$$\begin{aligned} &F(\psi(G(Ax,By,By)), \psi(G(Sx,Ty,Ty)), \psi(G(Sx,Sx,Ax)),\\ &\psi(G(Ty,By,By)), \psi(G(Sx,By,By)), \psi(G(Ax,Ty,Ty))) \le 0. \end{aligned}$$

Hence the conditions of Theorem 4.2 are satisfied and the conclusions of Theorem 5.3 follows.

Similarly, from Theorem 4.4 we obtain

Theorem 5.4 *Let A, B, S and T be self mappings of a G - metric space (X, G) such that*

$$\begin{aligned}&F\Big(\int_0^{G(Ax,Ax,By)}h(t)dt, \int_0^{G(Sx,Sx,By)}h(t)dt, \int_0^{G(Sx,Sx,Ax)}h(t)dt,\\&\int_0^{G(Ty,Ty,By)}h(t)dt, \int_0^{G(Sx,Sx,By)}h(t)dt, \int_0^{G(Ax,Ax,Ty)}h(t)dt\Big) \le 0,\end{aligned} \tag{5.2}$$

for all $x, y \in X$, where $F \in \mathcal{F}_G$ and $h(t)$ as in Theorem 5.1.

If (A,S) and (B,T) satisfy $CLR_{(S,T)}$ - property, then

i) $C(A,S) \neq \emptyset$,

ii) $C(B,T) \neq \emptyset$.

Moreover, if (A,S) and (B,T) are weakly compatible, then A, B, S and T have a unique common fixed point.

From Theorem 5.4 and Example 3.2 we obtain

Theorem 5.5 *Let (X, G) be a G - metric space and A, B, S and T be self mappings of X satisfying*

$$\begin{aligned}&\int_0^{G(Ax,Ax,By)}h(t)dt \le k\max\Big\{\int_0^{G(Sx,Sx,Ty)}h(t)dt, \int_0^{G(Sx,Sx,Ax)}h(t)dt,\\&\int_0^{G(Ty,Ty,By)}h(t)dt, \frac{\int_0^{G(Sx,Sx,By)}h(t)dt + \int_0^{G(Ax,Ax,Ty)}h(t)dt}{2}\Big\},\end{aligned}$$

for all $x, y \in X$, $k \in [0,1)$ and $h(t)$ as in Theorem 5.1.

If (A,S) and (B,T) satisfy $CLR_{(S,T)}$ - property, then

i) $C(A,S) \neq \emptyset$,

ii) $C(B,T) \neq \emptyset$.

Moreover, if (A,S) and (B,T) are weakly compatible, then A, B, S and T have a unique common fixed point.

Remark 5.1 *If $h(t) = 1$, from Theorem 5.5 we obtain Theorem 4.6.*

From Theorems 5.3, 5.4 and Examples 3.1 – 3.9 we obtain new particular results.

5.2. Fixed Points for Almost Contractive Mappings in G - Metric Spaces

Definition 5.1 *Let (X, d) be a metric space. A mapping $T : X \to X$ is called weak contractive* (Berinde, 2003; 2004) *or almost contractive* (Berinde, 2010) *if there exist $\delta \in (0,1)$ and some $L \ge 0$ such that*

$$d(Tx,Ty) \le \delta d(x,y) + Ld(y,Tx) \text{ for all } x, y \in X.$$

The following theorem is proved in (Berinde, 2010).

Theorem 5.6 (Berinde, 2010) *Let* (X,d) *be a metric space and* $T,S:X\to X$ *be mappings for which there exists* $a\in(0,1)$ *and some* $L\ge 0$ *such that*

$$d(Tx,Ty)\le ad(Sx,Sy)+Ld(Sy,Tx),$$

for all $x,y\in X$.

If $T(X)\subset S(X)$ *and* $S(X)$ *is a complete subspace of* X, *then* T *and* S *have a unique point of coincidence. Moreover, if* T *and* S *are weakly compatible, then* T *and* S *have a unique common fixed point.*

A similar result is obtained if

$$d(Tx,Ty)\le ad(Sx,Sy)+L\min\{d(Sx,Tx),d(Sy,Ty),d(Sx,Ty),d(Tx,Sy)\},$$

where $a\in(0,1)$ and $L\ge 0$.

In (Babu *et el.*, 2008), a similar result is obtained if

$$d(Tx,Ty)\le \delta m(x,y)+L\min\{d(Sx,Tx),d(Sy,Ty),d(Sx,Ty),d(Tx,Sy)\},$$

where $\delta\in(0,1)$, $L\ge 0$ and

$$m(x,y)=\max\{d(Sx,Sy),\frac{d(Tx,Sx)+d(Ty,Sy)}{2},\frac{d(Sx,Ty)+d(Tx,Sy)}{2}\}.$$

The following functions $F:\mathbb{R}_+^6\to\mathbb{R}$ satisfy conditions (F_1), (F_2) and (F_3).

Example 5.1 $F(t_1,...,t_6)=t_1-\delta\max\left\{t_2,\frac{t_3+t_4}{2},\frac{t_5+t_6}{2}\right\}-$ $-L\min\{t_3,t_4,t_5,t_6\}$, *where* $\delta\in(0,1)$ *and* $L\ge 0$.

Example 5.2 $F(t_1,...,t_6)=t_1-at_2-L\min\{t_3,t_4,t_5,t_6\}$, *where* $a\in(0,1)$ *and* $L\ge 0$.

Example 5.3 $F(t_1,...,t_6)=t_1-k\max\left\{t_2,t_3,t_4,\frac{t_5+t_6}{2}\right\}-$ $-L\min\{t_3,t_4,t_5,t_6\}$, *where* $k\in(0,1)$ *and* $L\ge 0$.

Example 5.4 $F(t_1,...,t_6)=t_1-k\max\{t_2,t_3,t_4,t_5,t_6\}-$ $-L\min\{t_3,t_4,t_5,t_6\}$, *where* $k\in(0,1)$ *and* $L\ge 0$.

Example 5.5 $F(t_1,...,t_6)=t_1-k\max\left\{t_2,\frac{t_3+t_4}{2},\frac{t_5+t_6}{2}\right\}-$ $L\min\{t_3,t_4,\sqrt{t_4t_5},\sqrt{t_5t_6}\}$, *where* $k\in(0,1)$ *and* $L\ge 0$.

Example 5.6 $F(t_1,...,t_6)=t_1-k\max\{t_2,t_3,\sqrt{t_4t_5},\sqrt{t_5t_6}\}-$ $L\min\{t_3,t_4,t_5,t_6\}$, *where* $k\in(0,1)$ *and* $L\ge 0$.

Example 5.7 $F(t_1,...,t_6)=t_1-\max\{k(t_3+t_4),k(t_5+t_6)\}-$

$L\min\{t_3,t_4,t_5,t_6\}$, *where* $k\in\left(0,\frac{1}{2}\right)$ *and* $L\geq 0$.

Example 5.8 $F(t_1,...,t_6)=t_1-\max\left\{t_2,\alpha t_3,\alpha t_4,\frac{\alpha(t_5+t_6)}{2}\right\}-$ $L\min\{t_3,t_4,t_5,t_6\}$, *where* $\alpha\in(0,1)$ *and* $L\geq 0$.

By Theorem 4.2 and Example 5.1 we obtain

Theorem 5.7 *Let* A,B,S *and* T *be self mappings of a* G *- metric space* (X,G) *such that*

$$\psi(G(Ax,By,By))\leq\delta\max\{\psi(G(Sx,Ty,Ty)),$$
$$\frac{\psi(G(Sx,Sx,Ax))+\psi(G(Ty,By,By))}{2},\frac{\psi(G(Sx,By,By))+\psi(G(Ax,Ty,Ty))}{2}\},$$

where $\delta\in(0,1)$, $L\geq 0$, *for all* $x,y\in X$ *and* ψ *is an almost altering distance.*

If (A,S) *and* (B,T) *satisfy* $CLR_{(S,T)}$ *- property, then*

i) $C(A,S)\neq\emptyset$,

ii) $C(B,T)\neq\emptyset$.

Moreover, if (A,S) *and* (B,T) *are weakly compatible, then* A,B,S *and* T *have a unique common fixed point.*

Theorem 5.8 *Let* A,B,S *and* T *be self mappings of a* G *- metric space* (X,G) *such that*

$$\int_0^{G(Ax,By,By)}h(t)dt\leq\delta\max\{\int_0^{G(Sx,Ty,Ty)}h(t)dt,$$
$$\frac{\int_0^{G(Sx,Sx,Ax)}h(t)dt+\int_0^{G(Ty,By,By)}h(t)dt}{2},\frac{\int_0^{G(Sx,By,By)}h(t)dt+\int_0^{G(Ax,Ty,Ty)}h(t)dt}{2}\}+$$
$$L\min\{\int_0^{G(Sx,Sx,Ax)}h(t)dt,\int_0^{G(Ty,By,By)}h(t)dt,\int_0^{G(Sx,By,By)}h(t)dt,\int_0^{G(Ax,Ty,Ty)}h(t)dt\},$$

where $\delta\in(0,1)$ *and* $L\geq 0$, *for all* $x,y\in X$ *and* $h(t)$ *as in Theorem 5.1.*

If (A,S) *and* (B,T) *satisfy* $CLR_{(S,T)}$ *- property, then*

i) $C(A,S)\neq\emptyset$,

ii) $C(B,T)\neq\emptyset$.

Moreover, if (A,S) *and* (B,T) *are weakly compatible, then* A,B,S *and* T *have a unique common fixed point.*

Remark 5.2 *Similar results are obtained by Examples 5.2 – 5.8.*

5.3. Fixed Points for Mappings Satisfying φ - Contractive Conditions in G - Metric Spaces

As in (Matkowski, 1997), let ϕ be the set of all real nondecreasing continuous functions $\varphi:[0,\infty)\to 0,\infty)$ with $\lim_{n\to\infty}\varphi^n(t)=0$.

If $\varphi\in\phi$, then

1) $\varphi(t)<t$ for all $t\in(0,\infty)$,
2) $\varphi(0)=0$.

The following functions $F:\mathbb{R}_+^6\to\mathbb{R}$ satisfy conditions (F_1), (F_2) and (F_3).

Example 5.9 $F(t_1,...,t_6)=t_1-\varphi(\max\{t_2,t_3,t_4,t_5,t_6\})$.

Example 5.10 $F(t_1,...,t_6)=t_1-\varphi\left(\max\left\{t_2,t_3,t_4,\dfrac{t_5+t_6}{2}\right\}\right)$.

Example 5.11 $F(t_1,...,t_6)=t_1-\varphi\left(\max\left\{t_2,\dfrac{t_3+t_4}{2},\dfrac{t_5+t_6}{2}\right\}\right)$.

Example 5.12 $F(t_1,...,t_6)=t_1-\varphi(\max\{t_2,\sqrt{t_3t_4},\sqrt{t_3t_5},\sqrt{t_4t_6},\sqrt{t_5t_6}\})$

Example 5.13 $F(t_1,...,t_6)=t_1-\varphi(at_2+bt_3+ct_4+dt_5+et_6)$, *where* $a,b,c,d,e\geq 0$ *and* $a+b+c+d+e<1$.

Example 5.14 $F(t_1,...,t_6)=t_1-\varphi\left(at_2+b\dfrac{\sqrt{t_5t_6}}{1+t_3+t_4}\right)$, *where* $a,b\geq 0$ *and* $a+b<1$.

Example 5.15

$F(t_1,...,t_6)=t_1-\varphi\left(at_2+b\max\{t_3,t_4\}+c\max\left\{\dfrac{t_3+t_4}{2},\dfrac{t_5+t_6}{2}\right\}\right)$, *where* $a,b,c\geq 0$ *and* $a+b+c<1$.

Example 5.16

$F(t_1,...,t_6)=t_1-\varphi\left(at_2+b\max\left\{\dfrac{2t_4+t_5}{3},\dfrac{2t_4+t_6}{3},\dfrac{t_3+t_5+t_6}{3}\right\}\right)$, *where* $a,b\geq 0$ *and* $a+b<1$.

By Theorem 4.2 and Example 5.9 we obtain

Theorem 5.9 *Let* A,B,S *and* T *be self mappings of a* G *- metric space* (X,G) *such that*

$$\psi(G(Ax,By,By))\leq\varphi(\max\{\psi(G(Sx,Ty,Ty)),\psi(G(Sx,Sx,Ax)),$$
$$\psi(G(Ty,By,By)),\psi(G(Sx,By,By)),\psi(G(Ax,Ty,Ty))\}),$$

for all $x, y \in X$, $\varphi \in \phi$ *and* ψ *is an almost altering distance.*

If (A,S) *and* (B,T) *satisfy* $CLR_{(S,T)}$ *- property, then*

i) $C(A,S) \neq \varnothing$,

ii) $C(B,T) \neq \varnothing$.

Moreover, if (A,S) *and* (B,T) *are weakly compatible, then* A, B, S *and* T *have a unique common fixed point.*

By Theorem 5.9 and Theorem 5.3 we obtain

Theorem 5.10 *Let* A, B, S *and* T *be self mappings of a* G *- metric space* (X,G) *such that*

$$\int_0^{G(Ax,By,By)} h(t)dt \leq \varphi(\max\{\int_0^{G(Sx,Ty,Ty)} h(t)dt, \int_0^{G(Sx,Sx,Ax)} h(t)dt,$$
$$\int_0^{G(Ty,By,By)} h(t)dt, \int_0^{G(Sx,By,By)} h(t)dt, \int_0^{G(Ax,Ty,Ty)} h(t)dt)$$

for all $x, y \in X$, $\varphi \in \phi$ *and* $h(t)$ *as in Theorem 5.1.*

If (A,S) *and* (B,T) *satisfy* $CLR_{(S,T)}$ *- property, then*

i) $C(A,S) \neq \varnothing$,

ii) $C(B,T) \neq \varnothing$.

Moreover, if (A,S) *and* (B,T) *are weakly compatible, then* A, B, S *and* T *have a unique common fixed point.*

Remark 5.3 *By Examples 5.10 – 5.16 we obtain similar results.*

If $\psi(t) = t$, by Theorem 5.9 we obtain

Theorem 5.11 *Let* A, B, S *and* T *be self mappings of a* G *- metric space* (X,G) *such that*

$$G(Ax, By, By) \leq \varphi(\max\{G(Sx, Ty, Ty), G(Sx, Sx, Ax),$$
$$G(Ty, By, By), G(Sx, By, By), G(Ax, Ty, Ty)\}),$$

for all $x, y \in X$ *and* $\varphi \in \phi$.

If (A,S) *and* (B,T) *satisfy* $CLR_{(S,T)}$ *- property, then*

i) $C(A,S) \neq \varnothing$,

ii) $C(B,T) \neq \varnothing$.

Moreover, if (A,S) *and* (B,T) *are weakly compatible, then* A, B, S *and* T *have a unique common fixed point.*

Corollary 5.1 (**Theorem 2.2** (Giniswamy and Maheshwari, 2014)) *Let* (X,G) *be a symmetric* G *- metric space and* A, B, S *and* T *four self mappings of* X *such that*

1) (A,S) *and* (B,T) *satisfy* $CLR_{(S,T)}$ *- property,*

2) $G(Ax, By, Bz) \leq \varphi(\max\{G(Sx, Ty, Tz), G(Sx, By, Bz), G(Ty, By, Bz), G(By, Ty, Tz)\})$,

for all $x, y, z \in X$ *and* $\varphi \in \phi$,

3) (A,S) *and* (B,T) *are weakly compatible.*

Then A,B,S *and* T *have a unique common fixed point.*

Proof. If $y=z$, by 2) we have

$$G(Ax,By,By)\le\varphi(\max\{G(Sx,Ty,Ty),G(Sx,By,By),G(Ty,By,By),G(By,Ty,Ty)\}).$$

Since (X,G) is symmetric and φ is non decreasing, then

$$\begin{aligned}G(Ax,By,By)&\le\varphi(\max\{G(Sx,Ty,Ty),G(Sx,By,By),G(Ty,Ty,By)\})\\&\le\varphi(\max\{G(Sx,Ty,Ty),G(Sx,Sx,Ax),G(Ty,By,By),\\&\quad G(Sx,By,By),G(Ax,Ty,Ty)\}),\end{aligned}$$

and by Theorem 5.11, A,B,S and T have a unique common fixed point.

5.4. Fixed Points for (φ,ψ) - Weakly Contractive Mappings in G - Metric Spaces

In 1997, Alber and Guerre-Delabriere (Alber and Guerre-Delabriere, 1997) defined the concept of weak contraction as a generalization of contraction and established the existence of fixed points for self mappings in Hilbert spaces. Rhoades (Rhoades, 2001) extended this concept in metric spaces. In (Beg and Abbas, 2006), the authors studied the existence of fixed points for a pair of (φ,ψ) - weakly compatible mappings.

New results are obtained in (Dorić, 2009; Raswan and Saleh, 2013) and in other papers.

The study of common fixed points of (φ,ψ) - weakly contractions with (EA) - property is initiated in (Sintunavarat and Kumam, 2011).

Also, some fixed point theorems for mappings with common limit range property satisfying (φ,ψ) - weakly contractive conditions are proved in (Imdad and Chauhan, 2013) and in other papers.

Definition 5.2

1) *Let* Ψ *be the set of all functions* $\psi:[0,\infty)\to 0,\infty)$ *satisfying*

$a)$ ψ *is continuous,*

$b)$ $\psi(0)=0$ *and* $\psi(t)>0$, $\forall t>0$.

2) *Let* Φ *be the set of all functions* $\phi:[0,\infty)\to 0,\infty)$ *satisfying*

$a)$ ϕ *is lower semi - continuous,*

$b)$ $\phi(0)=0$ *and* $\phi(t)>0$, $\forall t>0$.

The following functions $F:\mathbb{R}_+^6\to\mathbb{R}$ satisfy conditions $(F_1),(F_2)$ and (F_3).

Example 5.17 $F(t_1,...,t_6)=\psi(t_1)-\psi\left(\max\left\{t_2,t_3,t_4,\dfrac{t_5+t_6}{2}\right\}\right)+\phi(\max\{t_3,t_4,t_5,t_6\})$.

Example 5.18

$$F(t_1,...,t_6)=\psi(t_1)-\psi(\max\{t_2,t_3,t_4,t_5,t_6\})+\phi\left(\max\left\{t_2,t_3,t_4,\frac{t_5+t_6}{2}\right\}\right).$$

Example 5.19

$$F(t_1,...,t_6)=\psi(t_1)-\psi\left(\max\left\{t_2,\frac{t_3+t_4}{2},\frac{t_5+t_6}{2}\right\}\right)+\phi(\max\{t_2,t_3,t_4,t_5,t_6\}).$$

Example 5.20

$$F(t_1,...,t_6)=\psi(t_1)-\psi\left(\max\left\{t_2,\frac{t_3+t_4}{2},\frac{t_5+t_6}{2}\right\}\right)+\phi\left(\max\left\{t_2,t_3,t_4,\frac{t_5+t_6}{2}\right\}\right)$$

Example 5.21

$$F(t_1,...,t_6)=\psi(t_1)-\psi\left(\max\left\{t_2,t_3,t_4,\frac{t_5+t_6}{2}\right\}\right)+\phi(\max\{\sqrt{t_3t_6},\sqrt{t_2t_5},\sqrt{t_5t_6}\})$$

Example 5.22

$$F(t_1,...,t_6)=\psi(t_1)-\psi(\max\{\sqrt{t_3t_6},\sqrt{t_2t_5},\sqrt{t_4t_6}\})+\phi(\max\{t_2,t_3,t_4,t_5,t_6\})$$

Example 5.23

$$F(t_1,...,t_6)=\psi(t_1)-\psi\left(\frac{\sqrt{t_3t_6}+\sqrt{t_4t_5}+\sqrt{t_2t_6}}{1+\sqrt{t_3t_4}+\sqrt{t_4t_6}+\sqrt{t_2t_3}}\right)+\phi(\max\{t_2,t_3,t_4,t_5,t_6\})$$

By Theorem 4.3 and Example 5.17 we obtain

Theorem 5.12 *Let A, B, S and T be self mappings of a G - metric space (X, G) such that*

$$G(Ax,By,By)\le\psi(M_1(x,y))-\phi(M_2(x,y)),$$

for all $x, y \in X$, where

$$M_1(x,y)=\max\{G(Sx,Ty,Ty),G(Sx,Sx,Ax),$$
$$G(Ty,By,By),\frac{G(Sx,By,By)+G(Ax,Ty,Ty)}{2}\},$$

$$M_2(x,y)=\max\{G(Sx,Sx,Ax),G(Ty,Ty,By),G(Sx,By,By),G(Ax,Ty,Ty)\},$$

$\psi\in\Psi$ and $\varphi\in\phi$.

If (A,S) and (B,T) satisfy $CLR_{(S,T)}$ - property, then

i) $C(A,S)\neq\emptyset$,

ii) $C(B,T)\neq\emptyset$.

Moreover, if (A,S) and (B,T) are weakly compatible, then A, B, S and T have a unique common fixed point.

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TEOREME DE PUNCT FIX PENTRU DOUĂ PERECHI DE FUNCŢII CU PROPRIETATEA LIMITEI COMUNE ÎN SPAŢII *G* – METRICE

(Rezumat)

Scopul acestei lucrări este demonstrarea unei teoreme de punct fix pentru două perechi de funcţii în spaţii *G* - metrice, care să generalizeze rezultatele din (Popa și Patriciu, 2014) şi să unifice rezultatele din (Giniswamy și Maheshwari, 2014). De asemenea, este obţinut un rezultat nou pentru un şir de funcţii. În ultima parte a lucrării, ca aplicaţii, sunt obţinute câteva rezultate de punct fix pentru funcţii care satisfac o condiţie contractivă de tip integral, pentru funcţii aproape contractive, pentru funcţii ϕ – contractive şi (ϕ,ψ) – contractive în spaţii *G* – metrice.