BULETINUL INSTITUTULUI POLITEHNIC DIN IAŞI
Publicat de
Universitatea Tehnică „Gheorghe Asachi” din Iaşi
Volumul 63 (67), Numărul 2, 2017
Secţia
MATEMATICĂ. MECANICĂ TEORETICĂ. FIZICĂ

# DIMENSIONALITY AND NON-DIFFERENTIABILITY

BY

**IRINEL CASIAN BOTEZ***

"Gheorghe Asachi" Technical University of Iaşi,
Faculty of Electronics, Telecomunication and Information Technology

Received: June 16, 2017
Accepted for publication: September 26, 2017

**Abstract**. Dimensionality is one of principal characteristics that define the material parameters. The same compound can exhibit dramatic different properties depending on whether it is arranger in 1D, 2D or 3D structure. Although quasi-1D (*e.g.* nanotubes), 2D (*e.g.* grapheme) and, of course, 3D physical objects are well documented, dimensionality is conspicuously absent among the theoretical approach. We investigate the link between this dimensionality and the differentiable approach.

**Keywords:** non-differentiability; fractals.

## 1. The Theory

The present day physics, since the time of Newton and Leibniz, the founders of the integro-differential calculus, is based on the unjustified assumption of the differentiability of the space-time continuum. We say "unjustified" because there is neither a priori principle nor definite experiments that impose such approach. Moreover, this hypothesis is clearly broken by the quantum mechanical behavior. It was demonstrated by Feynman (Feynman and Hibbs, 1965) that the typical paths of quantum mechanical particles are continuous but non-differentiable.

---

*Corresponding author; *e-mail*: irinel_casian@yahoo.com

A geometrical intuitive notion of dimension is given by the exponential link between object's bulk and size (a linear distance) (Theiler, 1990):

$$bulk \sim size^{dimension} \tag{1}$$

So, if the space is differentiable, the *dimension* is a constant, the topological dimension of the object, $D_T$. If the space is continuous but non-differentiable, the *dimension* is different from the topological dimension:

$$dimension = D_T - D_F \tag{2}$$

and the bulk tends to infinity when the size tends to zero, which means, from Eq.(1), that dimension is negative:

$$D_T < D_F \tag{3}$$

Mandelbrot named $D_F$ the *fractal dimension* of the object (Mandelbrot, 1983). In other words, a non-differentiable space-time continuum is necessarily *fractal*, in the general meaning initially given by Mandelbrot (Mandelbrot, 1975; Mandelbrot, 1983).

In such systems implying fractals and non-differentiability, Scale Relativity Theory proposes the introducing of a new frame of thought where all scales co-exist simultaneously as different worlds, but are connected together via scale-differential equations. The quantum behavior may be reinterpreted as a manifestation of scale relativity.

A question is rising: what is the meaning of new differential in respect to scale? We must remember that Lobesgue's theorem (Titchmarsh, 1939) states that: *a continuous curve of finite length is differentiable or almost everywhere differentiable*.

Now let us consider the set of continuous real-valued functions $f(s)$, defined on a compact set $I$ of $\mathbb{R}$. We denote this set by $C^0(I)$ and the subset of continuous and nowhere differentiable on $I$ by $\mathcal{C}^0(I)$. The variable $s$, proper time, provides the parameterization of the graph, $\Gamma$, of the function $f(s)$. The classical way to construct an intrinsic coordinates system on the graph $\Gamma$ is to introduce the so-called *curvilinear coordinate* as the *length*, $\mathcal{L}(f;s,s_0)$, of the graph $\Gamma$ between the points $[s_0, f(s_0)]$ and $[s, f(s)]$, where $s_0 \in I$ is a given origin. If $f(s)$ is non-differentiable, we cannot use such construction because we have the converse of the Lebesgue's theorem: *if $f(s)$ is everywere non-differentiable then the length of $f(s)$ is infinite.*

In the framework of Scale Relativity Model is introduced a new point of view which is that we can have access:

– not to the function $f(s)$, but to a representation of it (controlled by a resolution constraint $\varepsilon$)

– to the behavior of this representation when this resolution changes.

This representation of the function $f(s)$ is a one-parameter family of real-valued functions, denoted $F(s,\varepsilon)$, $\varepsilon \in \mathbb{R}^+$, which has the properties that;

a) for all $\varepsilon > 0$, the function $F(s,\varepsilon)$ is differentiable;

b) $\lim\limits_{\varepsilon \to 0} F(s,\varepsilon) = f(s)$.

Such functions are named *fractal functions.* Fractal function $F(s,\varepsilon)$ is a function of two variables, $s$ (in space-time) and $\varepsilon$ (in scale space). A common fractal function is the length, $\mathfrak{L}(s,\varepsilon)$, of the polygonal approximation of the graph $\Gamma$ of a non-differentiable function, considered between the points $A = A_0\left[s_0, f(s_0)\right]$ and $B = \left[s, f(s)\right]$.

This length increases monotonically when the resolution $\varepsilon$ tends to zero. So the length of the graph of a continuous and almost everywhere non-differentiable function is *scale divergent*.

Therefore, this law of divergence, named scale law, was intensively studied (LeMehaute, 1991; Tricot, 1999; Cresson, 2002). For example, Cresson (Cresson, 2002) defines a *scale variable* as:

$$E = ln\left(\frac{\varepsilon}{\varepsilon_0}\right) \tag{4}$$

where $\varepsilon > 0$ is an absolute resolution described with respect to a given origin of resolution $\varepsilon_0$. In this *scale reference system,* we have $E = 0$ for $\varepsilon = \varepsilon_0$, $E > 0$ for $\varepsilon > \varepsilon_0$ and $E < 0$ for $\varepsilon < \varepsilon_0$.

*The scale law* (Cresson, 2002) is given by:

$$\frac{d\mathfrak{L}(s,E)}{dE} = A(\mathfrak{L}) \tag{5}$$

where $A : \mathbb{R} \to \mathbb{R}$. This law describes a relation between the space reference system and the scale reference system. The simplest law corresponds to the first degree polynom:

$$\frac{d\mathcal{L}(s,E)}{dE} = a + b\mathcal{L} \tag{6}$$

with $a,b$ are functions of $s$ in most general case, "a" having the dimension of length and "b" being dimensionless. In our approach we shall consider that $b(s) = b = const.$.

Eq. (6) is an ordinary differential equation of the first order which have the solution:

$$\mathcal{L}(s,E) = -\frac{a(s)}{b} + \left[\frac{a(s)}{b} + \mathcal{L}_0(s)\right]e^{bE} \tag{7}$$

where $\mathcal{L}_0(s) = \mathcal{L}(s,0)$. Eq. (7) can be rewritten as:

$$\mathcal{L}(s,\varepsilon) \approx \mathcal{L}_1(s)\eta(s)\left(\frac{\lambda_0}{\varepsilon}\right)^{-b(s)} \tag{8}$$

In this circumstances, let us remember that, following Mandelbrot (Mandelbrot, 1983), the length of a fractal curve satisfies (Mandelbrot, 1983), page 36, I quote:

$$L(\varepsilon) \approx \varepsilon^{1-D} \tag{9}$$

where $D$ is the *fractal dimension* which describe the dimensional discordance of the fractal sets (in our case the set is the curve). Therefore, if we compare Eq. (8) with Eq. (9), we have:

$$-b(s) = D_F - 1 \tag{10}$$

So Eq. (7) become:

$$\mathcal{L}(s,\varepsilon) = \mathcal{L}_1(s) + \mathcal{L}_1(s)\eta(s)\left(\frac{\lambda_0}{\varepsilon}\right)^{D_F - 1} \tag{11}$$

Let $X\left(s,\varepsilon\right)$ be one of the representation of the axes, we have:

$$X\left(s,\varepsilon\right)=x\left(s\right)+x\left(s\right)\eta_x\left(s\right)\left(\frac{\lambda_0}{\varepsilon}\right)^{D_F-1} \tag{12}$$

In the simplest case, we have to restrict to the case of self-similar fractal curves which have constant fractal dimension.

## 2. Conclusions

The main conclusions of this paper is that first consequence of the reducing dimensionality is the non-differentiability (fractality) of space-time. In such situation we can use a representation of the non-differentiable quantity which will depend compulsory of resolution.

## REFERENCES

Cresson J., *Scale Relativity Theory for One-Dimensional Non-Differentiable Manifolds*, Chaos Solitons & Fractals, **14**, *4*, 553-562 (2002).

Feynman R., Hibbs A., *Quantum Mechanics and Path Integrals*, New York, McGraw-Hill (1965).

LeMehaute A., *Fractal Geometries*, Boca Raton, CRC Press (1991).

Mandelbrot B.B., *Les Objets Fractales*, Paris, Flammarion (1975).

Mandelbrot B.B., *The Fractal Geometry of Nature*, New York, W.H. Freeman and Company (1983).

Theiler J., *Estimating Fractal Dimension*, J. Opt. Soc. Am. A, **7**, *6*, 1055-1073 (1990).

Titchmarsh E.C., *The Theory of Functions*, London, Oxford University Press (1939).

Tricot C., *Curbes et Dimension Fractale*, Berlin, Springer-Verlag (1999).

DIMENSIONALITATE ȘI NON-DIFERENȚIABILITATE

(Rezumat)

Dimensionalitatea este una dintre principalele caracteristici care definesc parametrii materialului. Același compus poate prezenta în mod dramatic diferite proprietăți, în funcție de faptul dacă este aranjat în structură 1D, 2D sau 3D. Deși obiectele cvasi-1D (de ex., Nanotuburile), 2D (de exemplu, grafimele) și, desigur, obiectele fizice 3D sunt bine documentate, dimensionalitatea este absentă în mod vizibil în abordarea teoretică. Investigăm legătura dintre această dimensionalitate și abordarea diferențiată.