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FRACTAL ENTANGLEMENT

BY

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Abstract. In this article we present a generalization of Schrödinger equation, which we named fractal Schrödinger equation, and an generalization of notion of entanglement: fractal entanglement.

Keywords: fractal wave function; entanglement; fractal entanglement.

1. Introduction

Structures of the nature can be assimilated to complex systems, taking into account both their functionality, as well as their structure (Remo Badii 1997; Mitchell, 2011). The models commonly used to study the dynamics of complex systems are based on the assumption, otherwise unjustified, of the differentiability of the physical quantities that describe it, such as density, momentum, energy etc. - for a mathematical model see (Hou Thomas, 2009; Deville and Gatski, 2012) and for some applications, see (Rabinovich and Kalman, 2008; Zhang *et al*., 2009). The success of differentiable models must be understood sequentially, *i.e*. on domains large enough that differentiability and integrability are valid.

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But differential method fails when facing the physical reality, with non-differentiable or non-integral physical dynamics, such as instabilities in the case of dynamics of complex structures instabilities that can generate both chaos and patterns.

In order to describe such physical dynamics of structures under severe plastic deformation, but still remaining tributary to a differential hypothesis, it is necessary to introduce, in an explicit manner, the scale resolution in the expressions of the physical variables that describe these dynamics and, implicitly, in the fundamental equations of “evolution” (for example, density, momentum and energy equations). This means that any dynamic variable, dependent, in a classical meaning, on both spatial coordinates and time (Batchelor, 2000), becomes, in this new context, dependent also on the scale resolution. In other words, instead of working with a dynamic variable, described through a strictly non-differentiable mathematical function, we will just work with different approximations of that function, derived through its averaging at different resolution scales. Consequently, any dynamic variable acts as the limit of a functions family, the function being non-differentiable for a null scale resolution and differentiable for a nonzero scale resolution.

This approach, well adapted for applications in the field of physics, where any real determination is conducted at a finite scale resolution, clearly implies the development both of a new geometric structure and of a physical theory (applied to complex structures) for which the motion laws, invariant to spatial and temporal coordinates transformations, are integrated with scale laws, invariant at scale transformations. Such a theory that includes the geometric structure based on the above presented assumptions was developed in the Scale Relativity Theory (Nottale, 2011) and more recently in the Extended Scale Relativity Theory (Agop *et al*., 2015). Both theories define the “fractal physics models” class (Nottale 1989; Nottale 2010; Nottale 2011; Naschie *et al*., 1995). In these models the differentiability in the dynamics of complex system is replaced by non-differentiability (fractality). Then the motions constrained on continuous, but differentiable curves in an Euclidian space are replaced with free motions, without any constrains, on fractal curves in a non-differentiable (fractal) space. Thus, the motion curves have double identity: trajectories of the fractal space and streamlines of a fractal fluid (Agop *et al*., 2015). In such conjecture, for time scale resolution that prove to be large when compared with the inverse of the highest Lyapunov exponent (Mandelbrot, 1983), the trajectories are replaced by “potential” trajectories, so that the concept of “definite positions” is substituted by that of “probability density”. Moreover, the complex system structural units (for example, the particles of a fluid) are substituted with the trajectories (geodesics) themselves so that any external constrains are interpreted as a selection of trajectories (geodesics) by means of measuring device.

2. Mathematical Model

Supposing that the motions of the structural units of the complex systems take place on fractal curves, the following consequences result (Nottale, 2011):

i) Any fractal curve of complex system structural units is explicitly scale resolution dependent δt. Its length becomes infinity when δt goes to zero;

We mention that a curve is a fractal if it satisfies the Lebesgue theorem (Mandelbrot, 1983), *i.e*. its length tends to infinity when the scale resolution becomes zero. Consequently, in this limit, a fractal curve is self-similar: every point reflect, the whole which can be translated into a property of holography (Mandelbrot, 1983).

ii) Through the substitution principle, δt will be identified with dt, *i.e*., $\delta t = dt$ so that, it will be considered as an independent variable.

iii) The dynamics of the structural units of the complex systems are described through fractal variables. Then, these variables are functions depending on both the space-time coordinates and the scale resolution since the infinitesimal time reflection invariance of any fractal variable is broken. So, in any point of the fractal curve, two derivatives of the variable field $Q(t, dt)$ are defined:

$$\begin{aligned} \frac{d_+Q(t,dt)}{dt} &= \lim_{\Delta t \to 0_+} \frac{Q(t+\Delta t, \Delta t) - Q(t, \Delta t)}{\Delta t} \\ \frac{d_-Q(t,dt)}{dt} &= \lim_{\Delta t \to 0_-} \frac{Q(t, \Delta t) - Q(t-\Delta t, \Delta t)}{\Delta t} \end{aligned} \tag{1}$$

The “+” sign corresponds to forward physical processes of complex system’s structural unit, while the “–” sign correspond to the backwards ones;

iv) The differential of the spatial coordinate $dX^i(t, dt)$ is expressed as the sum of the two differentials, *i.e*.:

$$d_\pm X^i(t,dt) = d_\pm x^i(t) + d_\pm \xi^i(t,dt); \tag{2}$$

The differential part, $d_\pm x^i(t)$, is scale resolution independent, while the fractal one, $d_\pm \xi^i(t)$, is scale resolution dependent.

v) The non-differentiable part of the spatial coordinate satisfies the equation (Mandelbrot, 1983):

$$d_{\pm}\xi^{i}\left(t,dt\right)=\lambda_{\pm}^{i}\left(dt\right)^{1/D_F} \tag{3}$$

where $\lambda_{\pm}^{i}$ are constant coefficients. By means of these coefficients, the fractalization type is specified, while by means of D_F the fractal dimension of the motion curves is defined.

In our opinion, the physical processes characterising both local and global properties of the complex systems imply dynamics on geodesics with various fractal dimensions. The diversity of the fractal dimensions of the geodesics have as a result the assimilation of the complex system with a multifractal (Mandelbrot, 1983). In such conjecture, for $D_F=2$, the complex system dynamics are described by quantum type processes. For $D_F<2$ the complex system dynamics are described by correlative type processes, while for $D_F>2$ the complex system dynamics are described by non-correlative type processes - for details see (Nottale, 1989; Naschie *et al*., 1995).

vi) The infinitesimal time reflection invariance of any fractal variable is recovered by summing the derivatives d_+/dt and d_-/dt in the non-differentiable operator (fractal operator):

$$\frac{\hat{d}}{dt}=\frac{1}{2}\left(\frac{d_+ + d_-}{dt}\right)-\frac{i}{2}\left(\frac{d_+ - d_-}{dt}\right) \tag{4}$$

This is the result of the Cresson's prolongation procedure applied to the dynamics of the complex system (Cresson, 2006). For example, the non-differentiable operator to the spatial coordinate yields the complex velocity field of the complex system.

$$\hat{V}^{i}=\frac{\hat{d}X^{i}}{dt}=V_D^{i}-V_F^{i} \tag{5}$$

with

$$V_D^{i}=\frac{1}{2}\frac{d_+X^{i}+d_-X^{i}}{dt},\quad V_F^{i}=\frac{1}{2}\frac{d_+X^{i}-d_-X^{i}}{dt} \tag{6}$$

The real part V_D^{i} of the complex velocity field is differentiable and scale resolution independent (we shall call it the differentiable velocity field), while the imaginary part of the complex velocity field, V_F^i , is non-differentiable and scale resolution dependent (we shall call it fractal velocity field);

vii) An infinite number of geodesics can be found relating any pair of points of a fractal manifold in the absence of any external constrain. Then, the infinity of geodesics in the bundle, together with their non-differentiability and the two values of the derivative (see Eqs. (1)-(3)) imply a description of the complex system structural units dynamics by means of a generalized statistical fluid dynamics (fractal fluid description). In such conjecture, the average values of the fractal variables must be considered in the previously mentioned sense. For example, the differential average values of the spatial coordinates is given by the relation:

$$\left\langle d_{\pm} X^{i} \right\rangle \equiv d_{\pm} x^{i} \tag{7}$$

with

$$\left\langle d_{\pm} \xi^{i} \right\rangle = 0 \tag{8}$$

The relation (8) implies that the differential average values of the spatial coordinates is null.

viii) The complex system dynamics can be described through a scale covariant derivative, the explicit form of which is obtained as follows. Let us consider that the motion fractal curves are immersed in a 3-dimensional space and that X^i are the spatial coordinate of a point on such a curve. We also consider the field $Q\left(X^i, t\right)$ and its Taylor's expansion up to the second order:

$$Q\left(X^{i}, t\right) = \partial_{t} Q dt + \partial_{i} Q dX^{i} + \frac{1}{2} \partial_{l} \partial_{k} Q dX^{l} dX^{k} \tag{9}$$

The functionalities of the relation (9) are valid in any point and more for the points X^i on the fractal curve which we have selected in (9). In these conditions, the forward and backward expressions of the field, Q , from (9) become

$$d_{\pm} Q = \partial_{t} Q dt + \partial_{i} Q d_{\pm} X^{i} + \frac{1}{2} \partial_{l} \partial_{k} Q d_{\pm} X^{l} d_{\pm} X^{k} \tag{10}$$

We assume that the average values of the all field Q and its derivatives coincide with themselves. Moreover, the differentials $d_{\pm}X^{i}$ and dt are independent. Then, the average of their products coincides with the product of averages. Consequently, (10) becomes

$$d_{\pm}Q=\partial_{t}Qdt+\partial_{i}Q\left\langle d_{\pm}X^{i}\right\rangle+\frac{1}{2}\partial_{l}\partial_{k}Q\left\langle d_{\pm}X^{l}d_{\pm}X^{k}\right\rangle \tag{11}$$

Even the differential average value of $d_{\pm}\xi^{i}$ is null, for the higher order of $d_{\pm}\xi^{i}$ the situation can still be different. Let us focus on the average values of the differentials $\left\langle d_{\pm}\xi^{l}d_{\pm}\xi^{k}\right\rangle$. Using (3) we can write:

$$\left\langle d_{\pm}\xi^{l}d_{\pm}\xi^{k}\right\rangle=\pm\lambda_{\pm}^{l}\lambda_{\pm}^{k}\left(dt\right)^{(2/D_{F})-1}dt \tag{12}$$

where we consider that the sign + corresponds to $dt>0$ and the sign – corresponds to $dt<0$

In these conditions, (11) takes the form:

$$d_{\pm}Q=\partial_{t}Qdt+\partial_{i}Qd_{\pm}X^{i}+\frac{1}{2}\partial_{l}\partial_{k}Qd_{\pm}x^{l}d_{\pm}x^{k}\pm\frac{1}{2}\partial_{l}\partial_{k}Q\left[\lambda_{\pm}^{l}\lambda_{\pm}^{k}\left(dt\right)^{(2/D_{F})-1}dt\right] \tag{13}$$

Multiplying the relation (13) by $\left(dt\right)^{-1}$ and neglecting the terms that contain differential factors we obtain (see the method from (Agop *et al*., 2015)):

$$\frac{d_{\pm}Q}{dt}=\partial_{t}Q+v_{\pm}^{i}\partial_{i}Q\pm\frac{1}{2}\lambda_{\pm}^{l}\lambda_{\pm}^{k}\left(dt\right)^{(2/D_{F})-1}\partial_{l}\partial_{k}Q \tag{14}$$

where

$$v_{\pm}^{i}=d_{\pm}x^{i}/dt$$

From here, the following operators can be defined:

$$\frac{d_{\pm}}{dt}=\partial_{t}+v_{\pm}^{i}\partial_{i}\pm\frac{1}{2}\lambda_{\pm}^{l}\lambda_{\pm}^{k}\left(dt\right)^{(2/D_{F})-1}\partial_{l}\partial_{k} \tag{15}$$

Now, taking into account (4), (5) and (15), let us calculate the fractal operator $\hat{d}/dt$. It results:

$$\frac{\hat{d}Q}{dt} = \partial_t Q + \hat{V}^i \partial_i Q + \frac{1}{4}(dt)^{(2/D_F)-1} D^{lk} \partial_l \partial_k Q \tag{16}$$

where

$$\begin{aligned} &D^{lk} = d^{lk} - i\bar{d}^{lk} \\ &d^{lk} = \lambda_+^l \lambda_+^k - \lambda_-^l \lambda_-^k, \quad \bar{d}^{lk} = \lambda_+^l \lambda_+^k + \lambda_-^l \lambda_-^k \end{aligned} \tag{17}$$

So, by means of the ralation (16) and (17) we can define the scale covariant derivative on the form:

$$\frac{\hat{d}}{dt} = \partial_t + \hat{V}^i \partial_i + \frac{1}{4}(dt)^{(2/D_F)-1} D^{lk} \partial_l \partial_k \tag{18}$$

Let us now consider the functionality of the following scale covariance principle: the physics laws are invariant with respect to scale transformations. In these conditions, the passage from the classical physics to the fractal physics can be implemented by replacing the time derivative d/dt by the fractal operator $\hat{d}/dt$. For example, applying the operator (18) to the complex velocity field (5), in the absence of any external constraint, the geodesics equation take the form:

$$\frac{\hat{d}\hat{V}^i}{dt} = \partial_t \hat{V}^i + \hat{V}^l \partial_l \hat{V}^i + \frac{1}{4}(dt)^{(2/D_F)-1} D^{lk} \partial_l \partial_k \hat{V}^i = 0 \tag{19}$$

It results that the acceleration, $\partial_t \hat{V}^i$, the convection, $\hat{V}^l \partial_l \hat{V}^i$, and the dissipation, $D^{lk} \partial_l \partial_k \hat{V}^i$, make their balance in any point of the motion fractal curve. The existence of the complex coefficient of viscosity-type $4^{-1}(dt)^{(2/D_F)-1} D^{lk}$ in the dynamics of the complex systems specifies that it is a rheological medium. So, it has memory.

For fractalisation by Markov type stochastic processes (Mandelbrot, 1983; Nottale, 2010), we have:

$$\lambda_+^i \lambda_+^l = \lambda_-^i \lambda_-^l = 2\lambda \delta^{il} \tag{20}$$

where δ^{il} is the Kronecker's pseudo-tensor.

Under these conditions, the geodesics equation takes the simple form:

$$\frac{\hat{d}\hat{V}^i}{dt} = \partial_t \hat{V}^i + \hat{V}^l \partial_l \hat{V}^i - i\lambda\left(dt\right)^{(2/D_F)-1} \partial^l \partial_l \hat{V}^i = 0 \ , \ i = 1,2,3 \tag{21}$$

or in vectorial form:

$$\frac{\hat{d}\hat{\boldsymbol{V}}}{dt} = \partial_t \hat{\boldsymbol{V}} + \left(\hat{\boldsymbol{V}} \cdot \nabla\right)\hat{\boldsymbol{V}} - i\lambda\left(dt\right)^{(2/D_F)-1} \Delta\hat{\boldsymbol{V}} = 0 \tag{22}$$

If the motion of the structural units in complex system is supposed irrotational, *i.e.* $\nabla\times\hat{\boldsymbol{V}} = \boldsymbol{0}$, we can choose $\hat{\boldsymbol{V}}$ of the form:

$$\hat{\boldsymbol{V}} = \nabla\phi \tag{23}$$

where ϕ is a complex velocity potential. Moreover, choosing this potential in the form:

$$\phi = -i\lambda\left(dt\right)^{(2/D_F)-1} \ln\psi \tag{24}$$

and substituting it in (22), it results:

$$\lambda\left(dt\right)^{(2/D_F)-1} \Delta\psi + i\partial_t\psi = 0 \tag{25}$$

up to an arbitrary phase factor which may be set to zero by a suitable choice of the phase of ψ (see details in (Agop *et al.*, 2015)).

For motions of the structural units of complex system on Peano's curves, $D_F = 2$, and at Compton scale, $\lambda = \hbar/2m_0$, with $\hbar$ the reduced Planck constant, $\hbar = h/2\pi$, and m_0 the rest mass of the entities of complex system, the relation (25) becomes the standard Schrödinger equation:

$$\frac{\hbar}{2m_0}\Delta\psi + i\frac{\partial\psi}{\partial t} = 0 \tag{26}$$

We conclude, in this stage, that the standard Schrödinger Eq. (26) is a particular case of a more general case represented by the Eq. (25), named in this paper *fractal Schrödinger equation*.

3. Entanglement

In 1935 Einstein, in a famous article (Einstein *et al*., 1935), proceeds from the principle that in a complete theory there must be an element that corresponds to each element of reality and demonstrates that in quantum mechanics the description of reality by its Schrödinger wave function is not complete.

In the same year, in another article, Schrödinger (Schrödinger, 1935) argues that starting from the same wave function interpretation which express the probability relationship between two separate daughter systems, we can introduce the notion of entanglement. By this, Schrödinger understands a new quantum state for each of the two systems, acquired by these once they were in contact. This new state of each system is maintained even after the two systems have been separated in space as much as possible. It has the character like that two states are twin, the modification of the state of one instantly causing the alteration of the state of the other, without being involved any energy transfer.

In 1952, Bohm (Bohm 1952a; Bohm 1952b) interprets this special quantum state as the expression of a hidden quantum variable, but in 1966, Bell (Bell, 1966) demonstrates that the existence of the hidden variable is impossible, and that it is only a new type of quantum state which have to be considered. The existence of this special quantum state can be demonstrated by checking some inequalities.

In 1993, Bennett (Bennett *et al*., 1993) shows that this kind of quantum state, the entanglement, can be used as an intrinsic encryption method of quantum information (information encoded using the quantum bits - qbits).

4. Conclusions

Taking into account the conclusion of paragraph 3 and the considerations in paragraph 4, we conclude that:

i) The existence of a fractal Schrödinger's equation, whose solution is the fractal wave function, allows us to introduce the generalizing notion of fractal entanglement.

ii) Fractal Entanglement is an intrinsic method of encrypting information transmitted by fractal bits.

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ENTANGLEMENT FRATAL

(Rezumat)

În acest articol prezentăm o generalizare a ecuației lui Schrödinger, pe care am numit-o ecuația fractală a lui Schrödinger, și o generalizare a noțiunii de entanglement: entanglement fractal.