EXISTENCE OF POSITIVE SOLUTIONS FOR A FRACTIONAL BOUNDARY VALUE PROBLEM

BY

RODICA LUCA*

“Gheorghe Asachi” Technical University of Iaşi, Department of Mathematics

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Abstract. We study the existence and multiplicity of positive solutions for a system of nonlinear Riemann-Liouville fractional differential equations with nonnegative and nonsingular nonlinearities, subject to multi-point boundary conditions which contain fractional derivatives.

Keywords: Riemann-Liouville fractional differential equations; multi-point boundary conditions; positive solutions; existence; multiplicity.

1. Introduction

Fractional differential equations describe many phenomena in several fields of engineering and scientific disciplines such as physics, biophysics, chemistry, biology (for example, the primary infection with HIV), economics, control theory, signal and image processing, thermoelasticity, aerodynamics, viscoelasticity, electromagnetics and rheology (Arafa et al., 2012; Baleanu et al., 2012; Cole, 1993; Das, 2008; Ding and Ye, 2009; Djordjevic et al., 2003; Ge and Ou, 2008; Kilbas et al., 2006; Klafter et al., 2011; Metzler and Klafter, 2000; Ostoja-Starzewski, 2007; Podlubny, 1999; Povstenko, 2015; Sabatier et al., 2007; Samko et al., 1993; Sokolov et al., 2002). Fractional differential

*Corresponding author; e-mail: rluca@math.tuiasi.ro
equations are also regarded as a better tool for the description of hereditary properties of various materials and processes than the corresponding integer order differential equations.

We consider the system of nonlinear ordinary fractional differential equations

\[
\begin{align*}
\left\{ 
D_0^+\alpha u(t) + f(t, u(t), v(t)) = 0, & \quad t \in (0,1), \\
D_0^+\beta v(t) + g(t, u(t), v(t)) = 0, & \quad t \in (0,1),
\end{align*}
\]

with the multi-point boundary conditions

\[
\begin{align*}
u^{(j)}(0) = 0, & \quad j = 0, ..., n - 2; \\
D_0^+\alpha u(t)|_{t=\xi_j} = \sum_{i=1}^{N} a_i D_0^+\alpha u(t)|_{t=\xi_j}, & \\
v^{(j)}(0) = 0, & \quad j = 0, ..., m - 2; \\
D_0^+\alpha v(t)|_{t=\eta_j} = \sum_{i=1}^{M} b_i D_0^+\alpha v(t)|_{t=\eta_j},
\end{align*}
\]

where \(\alpha, \beta \in \mathbb{R}, \alpha \in (n - 1, n], \beta \in (m - 1, m], n, m \in \mathbb{N}, n, m \geq 3, p_1, p_2, q_1, q_2 \in \mathbb{R}, p_1 \in [1, n - 2], p_2 \in [1, m - 2], q_1 \in [0, p_1], q_2 \in [0, p_2], \xi_i, \eta_i \in \mathbb{R} \) for all \(i = 1, ..., N (N \in \mathbb{N}), 0 < \xi_1 < \cdots < \xi_N \leq 1, \eta_1, \eta_i \in \mathbb{R} \) for all \(i = 1, ..., M (M \in \mathbb{N}), 0 < \eta_1 < \cdots < \eta_M \leq 1, \) and \(D_0^+\alpha \) denotes the Riemann-Liouville derivative of order \(k\) for \(k = \alpha, \beta, p_1, p_2, q_1, q_2\).

Under sufficient conditions on the nonnegative and nonsingular functions \(f\) and \(g\), we study the existence and multiplicity of positive solutions of problem \((S)-(BC)\). We use some theorems from the fixed point index theory (Aman, 1976; Zhou and Xu, 2006). By a positive solution of problem \((S)-(BC)\) we mean a pair of functions \((u, v) \in C([0,1], [0, \infty]))^2\) satisfying \((S)\) and \((BC)\) with \(u(t) > 0\) for all \(t \in (0,1)\) or \(v(t) > 0\) for all \(t \in (0,1)\).

The system \((S)\) with some positive parameters, subject to the boundary conditions \((BC)\) was investigated in (Henderson et al., 2017). The system \((S)\) with \(f(t, u, v) = f(t, v), g(t, u, v) = g(t, u)\) has been studied in (Henderson and Luca, 2017c). In this last paper, the authors use some different operators and different assumptions than those we use in this paper. The existence of positive solutions of the system \((S)\) with the coupled multi-point boundary conditions

\[
\begin{align*}
u^{(j)}(0) = 0, & \quad j = 0, ..., n - 2; \\
D_0^+\alpha u(t)|_{t=\xi_j} = \sum_{i=1}^{N} a_i D_0^+\alpha u(t)|_{t=\xi_j}, & \\
v^{(j)}(0) = 0, & \quad j = 0, ..., m - 2; \\
D_0^+\alpha v(t)|_{t=\eta_j} = \sum_{i=1}^{M} b_i D_0^+\alpha v(t)|_{t=\eta_j},
\end{align*}
\]

was studied in (Henderson and Luca, 2017b). For other papers which investigate the existence, nonexistence and multiplicity of positive solutions for systems of fractional differential equations with nonnegative or sign-changing nonlinearities, subject to various nonlocal boundary conditions we mention (Henderson and Luca, 2014a, b; Luca and Tudorache, 2014; Henderson and Luca, 2015; Henderson et al., 2015; Henderson and Luca, 2016a, b).

The paper is organized as follows. In Section 2, we present some auxiliary results which investigate a nonlocal boundary value problem for fractional differential equations, and we give the properties of the Green
functions associated to our problem. Section 3 contains the existence and multiplicity results for the positive solutions of problem (S)-(BC).

2. Auxiliary Results

We present here some auxiliary results from (Henderson and Luca, 2017a) that will be used to prove our main results.

We consider the fractional differential equation

\( D_0^\alpha u(t) + x(t) = 0, \quad 0 < t < 1, \) (1)

with the multi-point boundary conditions

\[ u^{(j)}(0) = 0, \quad j = 0, ..., n - 2; \quad D_0^{\alpha_i} u(t)|_{t=\xi_i} = \sum_{i=1}^{N} a_i D_0^{\alpha_i} u(t)|_{t=\xi_i}, \]

where \( \alpha \in (n - 1, n], n \in \mathbb{N}, n \geq 3, \alpha_i, \xi_i \in \mathbb{R}, i = 1, ..., N \ (N \in \mathbb{N}), \)

\( 0 < \xi_1 < \cdots < \xi_N \leq 1, \quad p_i, q_i \in \mathbb{R}, \quad p_i \in [1, n - 2], \quad q_i \in [0, p_i], \) and

\( x \in C(0,1) \cap L^1(0,1). \)

We denote by \( \Delta_1 = \frac{\Gamma(\alpha)}{\Gamma(\alpha-p_i)} - \frac{\Gamma(\alpha)}{\Gamma(\alpha-q_i)} \sum_{i=1}^{N} a_i \xi_i^{\alpha-q_i-1}. \)

**Lemma 1.** If \( \Delta_1 \neq 0 \), then the function \( u \in C[0,1] \) given by

\[ u(t) = \int_0^1 G_i(t, s)x(s)ds, \quad t \in [0,1], \]

is solution of problem (1)-(2), where

\[ G_i(t, s) = g_1(t, s) + \frac{t^{\alpha-1}}{\Delta_1} \sum_{i=1}^{N} a_i g_2(\xi_i, s), \quad \forall \ (t, s) \in [0,1] \times [0,1], \]

and

\[ g_1(t, s) = \begin{cases} \frac{1}{\Gamma(\alpha)}(t^\alpha(1-s)^{\alpha-p_i-1} - (t-s)^{\alpha-1}), & 0 \leq s \leq t \leq 1, \\ t^\alpha(1-s)^{\alpha-p_i-1}, & 0 \leq t \leq s \leq 1, \end{cases} \]

\[ g_2(t, s) = \begin{cases} \frac{1}{\Gamma(\alpha-q_i)}(t^{\alpha-q_i}(1-s)^{\alpha-p_i-1} - (t-s)^{\alpha-q_i-1}), & 0 \leq s \leq t \leq 1, \\ t^{\alpha-q_i}(1-s)^{\alpha-p_i-1}, & 0 \leq t \leq s \leq 1. \end{cases} \]

**Lemma 2.** The functions \( g_1 \) and \( g_2 \) given by (5) have the properties:

a) \( g_1(t, s) \leq h_1(s) \) for all \( t, s \in [0,1] \), where

\[ h_1(s) = \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-p_i-1}(1-(1-s)^{p_i}), \quad s \in [0,1]; \]

b) \( g_1(t, s) \geq t^{\alpha-1}h_1(s) \) for all \( t, s \in [0,1] \);

c) \( g_1(t, s) \leq \frac{t^{\alpha-1}}{\Gamma(\alpha)} \) for all \( t, s \in [0,1] \);

d) \( g_2(t, s) \geq t^{\alpha-q_i-1}h_2(s) \) for all \( t, s \in [0,1] \), where

\[ h_2(s) = \frac{1}{\Gamma(\alpha-q_i)}(1-s)^{\alpha-p_i-1}(1-(1-s)^{p_i-q_i}), \quad s \in [0,1]; \]

e) \( g_2(t, s) \leq \frac{t^{\alpha-q_i-1}}{\Gamma(\alpha-q_i)} \) for all \( t, s \in [0,1] \);
The functions $g_1$ and $g_2$ are continuous on $[0,1] \times [0,1]$; $g_1(t,s) \geq 0$, $g_2(t,s) \geq 0$ for all $t, s \in [0,1]$; $g_1(t,s) > 0$, $g_2(t,s) > 0$ for all $t, s \in (0,1)$.

**Lemma 3.** Assume that $a_i \geq 0$ for all $i = 1,...,N$ and $\Delta_1 > 0$. Then the function $G_1$ given by (4) is a nonnegative continuous function on $[0,1] \times [0,1]$ and satisfies the inequalities:

a) $G_1(t,s) \leq J_1(s)$ for all $t, s \in [0,1]$, where

$$J_1(s) = h_1(s) + \frac{1}{\Delta_1} \sum_{i=1}^{N} a_i g_2(\xi_i, s), s \in [0,1];$$

b) $G_1(t,s) \geq t^{\alpha-1} J_1(s)$ for all $t, s \in [0,1]$;

c) $G_1(t,s) \leq \sigma_1 t^{\alpha-1}$, for all $t, s \in [0,1]$, where

$$\sigma_1 = \frac{1}{\Gamma(\alpha)} + \frac{1}{\Delta_1 \Gamma(\alpha - q_1)} \sum_{i=1}^{N} a_i \xi_i^{\alpha - q_1 - 1}.$$

**Lemma 4.** Assume that $a_i \geq 0$ for all $i = 1,...,N$, $\Delta_1 > 0$, $x \in C(0,1) \cap L^1(0,1)$ and $x(t) \geq 0$ for all $t \in (0,1)$. Then the solution $u$ of problem (1)-(2) given by (3) satisfies the inequality $u(t) \geq t^{\alpha-1} u(t')$ for all $t, t' \in [0,1]$.

We can also formulate similar results as Lemmas 1-4 for the fractional boundary value problem

$$D_{0+}^{\beta} v(t) + y(t) = 0, \quad 0 < t < 1,$$

$$v^{(j)}(0) = 0, \quad j = 0,..,m-2; \quad D_{0+}^{\beta} v(t)|_{t=1} = \sum_{i=1}^{M} b_i D_{0+}^{\beta} v(t)|_{t=\eta_i},$$

where $\beta \in (m - 1, m], m \in \mathbb{N}, m \geq 3, b_i, \eta_i \in \mathbb{R}, i = 1, ..., M$ ($M \in \mathbb{N}$),

$0 < \eta_1 < \cdots < \eta_M \leq 1, \quad p_2, q_2 \in \mathbb{R}, \quad p_2 \in [1,m-2], \quad q_2 \in [0,p_2]$ and $y \in C(0,1) \cap L^1(0,1)$.

We denote by $\Delta_2, g_3, g_4, G_2, h_3, h_4, f_2$ and $\sigma_2$ the corresponding constants and functions for problem (6)-(7) defined in a similar manner as $\Delta_1, g_1, g_2, G_1, h_1, h_2, f_1$ and $\sigma_1$, respectively. More precisely, we have

$$\Delta_2 = \frac{\Gamma(\beta)}{\Gamma(\beta - p_2)} - \frac{\Gamma(\beta)}{\Gamma(\beta - q_2)} \sum_{i=1}^{M} b_i \eta_i^{\beta - q_2 - 1},$$

$$g_3(t,s) = \frac{1}{\Gamma(\beta)} \begin{cases} t^{\beta-1} (1 - s)^{\beta - p_2 - 1} - (t - s)^{\beta - 1}, & 0 \leq s \leq t \leq 1, \\ t^{\beta-1} (1 - s)^{\beta - p_2 - 1}, & 0 \leq t \leq s \leq 1, \end{cases}$$

$$g_4(t,s) = \frac{1}{\Gamma(\beta - q_2)} \begin{cases} t^{\beta-2} (1 - s)^{\beta - q_2 - 1} - (t - s)^{\beta - q_2 - 1}, & 0 \leq s \leq t \leq 1, \\ t^{\beta-2} (1 - s)^{\beta - q_2 - 1}, & 0 \leq t \leq s \leq 1, \end{cases}$$

$$G_2(t,s) = g_3(t,s) + \frac{t^{\beta-1}}{\Delta_2} \sum_{i=1}^{M} b_i g_4(\eta_i, s), \quad \forall (t,s) \in [0,1] \times [0,1],$$

$$h_3(s) = \frac{1}{\Gamma(\beta)} (1 - s)^{\beta - p_2 - 1} (1 - (1-s)^{p_2}), \quad s \in [0,1].$$
\[ h_4(s) = \frac{1}{\Gamma(\beta - q_2)}(1 - s)^{\beta - p_2 - 1}(1 - (1 - s)^{p_2 - q_2}), \quad s \in [0,1], \]

\[ J_2(s) = h_3(s) + \frac{1}{\Delta_2} \sum_{i=1}^{M} b_i g_4(\eta_i, s), \quad s \in [0,1], \]

\[ \sigma_2 = \frac{1}{\Gamma(\beta)} + \frac{1}{\Delta_2 \Gamma(\beta - q_2)} \sum_{i=1}^{M} b_i \eta_i^{\beta - q_2 - 1}. \]

The inequalities from Lemmas 3 and 4 for the functions \( G_2 \) and \( v \) are the following \( G_2(t, s) \leq J_2(s) \). \( G_2(t, s) \geq t^{\beta - 1} J_2(s) \), \( G_2(t, s) \leq \sigma_2 t^{\beta - 1} \), for all \( t, s \in [0,1] \), and \( v(t) \geq t^{\beta - 1} v(t') \) for all \( t, t' \in [0,1] \).

**Remark 1.** Under the assumptions of Lemma 4, and of the corresponding lemma for problem (6)-(7), for \( c \in (0,1) \), the solutions \( u, v \) of problems (1)-(2) and (6)-(7), respectively, satisfy the inequalities

\[ \min_{t \in [c,1]} u(t) \geq c^{\alpha - 1} \max_{t \in [0,1]} u(t), \]

\[ \min_{t \in [c,1]} v(t) \geq c^{\beta - 1} \max_{t \in [0,1]} v(t). \]

The proofs of our results are based on the following fixed point index theorems. Let \( E \) be a real Banach space, \( P \subset E \) a cone, “\( \preceq \)” the partial ordering defined by \( P \) and \( \theta \) the zero element in \( E \). For \( q > 0 \), let \( B_q = \{ u \in E, \| u \| < q \} \) be the open ball of radius \( q \) centered at \( \theta \), and its boundary \( \partial B_q = \{ u \in E, \| u \| = q \} \).

**Theorem 1.** (Amann, 1976) Let \( A: \overline{B}_q \cap P \to P \) be a completely continuous operator which has no fixed point on \( \partial B_q \cap P \). If \( \| Au \| \leq \| u \| \) for all \( u \in \partial B_q \cap P \), then \( i(A, B_q \cap P, P) = 1 \).

**Theorem 2.** (Amann, 1976) Let \( A: \overline{B}_q \cap P \to P \) be a completely continuous operator. If there exists \( u_0 \in P \setminus \{ \theta \} \) such that \( u - Au \neq \lambda u_0 \), for all \( \lambda \geq 0 \) and \( u \in \partial B_q \cap P \), then \( i(A, B_q \cap P, P) = 0 \).

**Theorem 3.** (Zhou and Xu, 2006) Let \( A: \overline{B}_q \cap P \to P \) be a completely continuous operator which has no fixed point on \( \partial B_q \cap P \). If there exists a linear operator \( L: P \to P \) and \( u_0 \in P \setminus \{ \theta \} \) such that

i) \( u_0 \leq Lu_0 \),

ii) \( Lu \leq Au \), \( \forall u \in \partial B_q \cap P \),

then \( i(A, B_q \cap P, P) = 0 \).

3. Main Results

In this section we investigate the existence and multiplicity of positive solutions for problem (S)-(BC) under various assumptions on the functions \( f \) and \( g \).
We present the assumptions that we shall use in the sequel.

\begin{enumerate}
\item[(H1)] \( \alpha, \beta \in \mathbb{R}, \ \alpha \in (n-1, n], \ \beta \in (m-1, m], \ n, m \in \mathbb{N}, \ n, m \geq 3, \)
\( p_1, p_2, q_1, q_2 \in \mathbb{R}, \ p_1 \in [1, n-2], \ p_2 \in [1, m-2], \ q_1 \in [0, p_1], \ q_2 \in [0, p_2], \)
\( \xi_1 \in \mathbb{R}, \ a_i \geq 0 \) for all \( i = 1, ..., N (N \in \mathbb{N}), \ 0 < \xi_1 < \cdots < \xi_N \leq 1, \eta_i \in \mathbb{R}, \)
\( b_i \geq 0 \) for all \( i = 1, ..., M (M \in \mathbb{N}), \ 0 < \eta_1 < \cdots < \eta_M \leq 1, \)
\( \Delta_1 = \frac{\Gamma(\alpha)}{\Gamma(\alpha - p_1)} - \frac{\Gamma(\beta)}{\Gamma(\beta - p_2)} \sum_{i=1}^{N} a_i \xi_i^{\alpha - q_1 - 1} > 0, \)
\( \Delta_2 = \frac{\Gamma(\beta)}{\Gamma(\beta - q_2)} \sum_{i=1}^{M} b_i \eta_i^\beta - q_2 - 1 > 0. \)
\end{enumerate}

\begin{enumerate}
\item[(H2)] The functions \( f, g : [0,1] \times [0, \infty) \times [0, \infty) \to [0, \infty) \) are continuous.
\end{enumerate}

By using Lemma 2, a solution of the following nonlinear system of integral equations
\[
\begin{align*}
    u(t) &= \int_0^t G_1(t, s)f(s, u(s), v(s)) \, ds, \quad t \in [0,1], \\
    v(t) &= \int_0^t G_2(t, s)g(s, u(s), v(s)) \, ds, \quad t \in [0,1]
\end{align*}
\]
is solution of problem (S)-(BC).

We consider the Banach space \( X = C[0,1] \) with supremum norm \( \| \cdot \| \) and the Banach space \( Y = X \times X \) with the norm \( \| (u, v) \|_X = \| u \| + \| v \| \). We define the cone \( P \subset Y \) by \( P = \{(u, v) \in Y, \ u(t) \geq 0, \ v(t) \geq 0 \) for all \( t \in [0,1] \}\).

We introduce the operators \( Q_1, Q_2 : Y \to X \) and \( Q : Y \to Y \) defined by
\[
    Q_1(u, v)(t) = \int_0^t G_1(t, s)f(s, u(s), v(s)) \, ds, \quad t \in [0,1],
\]
\[
    Q_2(u, v)(t) = \int_0^t G_2(t, s)g(s, u(s), v(s)) \, ds, \quad t \in [0,1],
\]
and \( Q(u, v) = (Q_1(u, v), Q_2(u, v)), (u, v) \in Y \).

Under the assumptions (H1) and (H2), it is easy to see that operator \( Q : P \to P \) is completely continuous. It is obvious that if \( (u, v) \) is a fixed point of operator \( Q \), then \( (u, v) \) is a solution of problem (S)-(BC). Therefore, we will study the existence and multiplicity of fixed points of operator \( Q \).

**Theorem 4.** Assume that (H1) and (H2) hold. If the functions \( f \) and \( g \) also satisfy the conditions
\begin{enumerate}
\item[(H3)] There exist \( p \geq 1 \) and \( q \geq 1 \) such that
\[
    f_0^p = \lim_{u \to +\infty} \sup_{v \geq 0} t \in [0,1] \frac{f(t, u, v)}{(u+v)^p} = 0 \quad \text{and} \quad g_0^q = \lim_{u \to +\infty} \sup_{v \geq 0} t \in [0,1] \frac{g(t, u, v)}{(u+v)^q} = 0;
\]
\item[(H4)] There exists \( c \in (0,1) \) such that
\[
    f_\infty = \lim_{u \to +\infty} \inf_{t \in [c,1]} \frac{f(t, u, v)}{u+v} = \infty \quad \text{and} \quad g_\infty = \lim_{u \to +\infty} \inf_{t \in [c,1]} \frac{g(t, u, v)}{u+v} = \infty,
\]
\end{enumerate}

then problem (S)-(BC) has at least one positive solution \( (u(t), v(t)) \), \( t \in [0,1] \).
Proof. For $c$ given in (H4) we define the cone
\[ P_0 = \left\{ (u, v) \in P, \min_{t \in [c, 1]} u(t) \geq c^{a-1} \|u\|, \min_{t \in [c, 1]} v(t) \geq c^{\beta-1} \|v\) \right\}. \]
From our assumptions and Remark 1, for any $(u, v) \in P$, we deduce that $Q(u, v) = (Q_1(u, v), Q_2(u, v)) \in P_0$, that is $Q(P) \subset P_0$.

We consider the functions $u_0, v_0: [0, 1] \to \mathbb{R}$ defined by
\[ u_0(t) = \int_0^t G_1(t, s) ds, \quad v_0(t) = \int_0^t G_2(t, s) ds, \quad t \in [0, 1], \]
that is $(u_0, v_0)$ is solution of problem (1)-(2) with $x(t) = x_0(t), y(t) = y_0(t), x_0(t) = 1, y_0(t) = 1$ for all $t \in [0, 1]$. Hence $(u_0, v_0) = Q(x_0, y_0) \in P_0$.

We define the set
\[ \tilde{M} = \{(u, v) \in P, \]

there exists $\lambda \geq 0$ such that $(u, v) = Q(u, v) + \lambda(u_0, v_0)$.\]

We will show that $\tilde{M} \subset P_0$ and $\tilde{M}$ is a bounded set of $Y$. If $(u, v) \in \tilde{M}$, then there exists $\lambda \geq 0$ such that $(u, v) = Q(u, v) + \lambda(u_0, v_0)$ or equivalently
\[
\begin{align*}
&u(t) = \int_0^1 G_1(t, s)(f(s, u(s), v(s)) + \lambda) ds, \quad t \in [0, 1], \\
v(t) = \int_0^1 G_2(t, s)(g(s, u(s), v(s)) + \lambda) ds, \quad t \in [0, 1].
\end{align*}
\]

By Remark 1, we obtain $(u, v) \in P_0$, so $\tilde{M} \subset P_0$, and
\[
\|u\| \leq \frac{1}{c^{a-1}} \min_{t \in [c, 1]} u(t), \quad \|v\| \leq \frac{1}{c^{\beta-1}} \min_{t \in [c, 1]} v(t), \quad \forall (u, v) \in \tilde{M}. \quad (8)
\]

From (H4) we have $f^{1, i}_1 = \infty$ and $g^{1, i}_1 = \infty$. Then for $\varepsilon_1 = \frac{2}{c^{a-1}m_1} > 0$,
\[
\varepsilon_2 = \frac{2}{c^{\beta-1}m_2} > 0, \text{ there exist } C_1 > 0, \ C_2 > 0 \text{ such that}
\]
\[
f(t, u, v) \geq \varepsilon_1 (u + v) - C_1, \quad g(t, u, v) \geq \varepsilon_2 (u + v) - C_2, \quad (9)
\]

\[
\forall (t, u, v) \in [c, 1] \times [0, \infty) \times [0, \infty),
\]

where $m_i = \int_c^1 j_i(s) ds$ and $j_i$, $i = 1, 2$ are defined in Lemma 3.

For $(u, v) \in \tilde{M}$ and $t \in [c, 1]$, by using Lemma 3 and relations (9), we obtain
\[
u(t) = Q_2(u, v)(t) + \lambda v_0(t) \geq Q_2(u, v)(t)
\]
\[
= \int_0^1 G_2(t, s) g(s, u(s), v(s)) ds \geq \int_c^1 \varepsilon_2 j_2(s) g(s, u(s), v(s)) ds
\]
\[
\geq c^{\beta-1} \varepsilon_2 j_2(1) \left[ \varepsilon_2 f(u(s) + v(s)) - C_2 \right] ds
\]
\[
\geq 2 \varepsilon_2 \min_{s \in [c, 1]} u(s) \geq C_3, \quad C_3 = c^{\beta-1} \varepsilon_2 j_2(1) C_1,
\]
and
\[
u(t) = Q_2(u, v)(t) + \lambda v_0(t) \geq Q_2(u, v)(t)
\]
\[
= \int_0^1 G_2(t, s) g(s, u(s), v(s)) ds \geq \int_c^1 \varepsilon_2 j_2(s) g(s, u(s), v(s)) ds
\]
\[
\geq c^{\beta-1} \varepsilon_2 j_2(1) \left[ \varepsilon_2 f(u(s) + v(s)) - C_2 \right] ds
\]
\[
\geq 2 \varepsilon_2 \min_{s \in [c, 1]} u(s) \geq C_3, \quad C_3 = c^{\beta-1} \varepsilon_2 j_2(1) C_1,
\]
\[ \geq c^{\beta-1} \varepsilon_2 m_2 \min_{t \in [c, 1]} v(t) - c^{\beta-1} m_2 C_2 \]
\[ \geq 2 \min_{t \in [c, 1]} v(t) - C_4, \quad C_4 = c^{\beta-1} m_2 C_2. \]

Therefore, we deduce
\[ \min_{t \in [c, 1]} u(t) \leq C_3, \quad \min_{t \in [c, 1]} v(t) \leq C_4, \quad \forall \ (u, v) \in \overline{M}. \]  

(10)

Now from relations (8) and (10), we obtain
\[ \|u\| \leq \frac{C_3}{c^{\alpha-1}}, \quad \|v\| \leq \frac{C_4}{c^{\beta-1}}, \]
\[ \|(u, v)\|_Y = \|u\| + \|v\| \leq \frac{C_3}{c^{\alpha-1}} + \frac{C_4}{c^{\beta-1}} = C_5, \]
for all \ (u, v) \in \overline{M}, that is \overline{M} is a bounded set of Y.

Besides, there exists a sufficiently large \( R_1 > 1 \) such that \( (u, v) \neq Q(u, v) + \lambda (u_0, v_0), \ \forall \ (u, v) \in \partial B_{R_1} \cap P, \ \forall \ \lambda \geq 0. \)

From (Amann, 1976), we deduce that the fixed point index of operator \( Q \) over \( B_{R_1} \cap P \) with respect to \( P \) is
\[ i(Q, B_{R_1} \cap P, P) = 0. \]  

(11)

Next, from assumption (H3), we conclude that for \( \varepsilon_3 = \frac{1}{4M_1} > 0 \) and \( \varepsilon_4 = \frac{1}{4M_2} > 0 \), there exists \( r_1 \in (0, 1] \) such that
\[ f(t, u, v) \leq \varepsilon_3 (u + v)^p, \quad g(t, u, v) \leq \varepsilon_4 (u + v)^q, \]
\[ \forall t \in [0, 1], \ u, v \geq 0, \ u + v \leq r_1, \]
where \( M_i = \int_0^1 f_i(s) ds, \ i = 1, 2. \)

By using (12), we deduce that for all \( (u, v) \in \overline{B}_{r_1} \cap P \) and \( t \in [0, 1] \)
\[ Q_1(u, v)(t) \leq \int_0^1 f_1(s) \varepsilon_3 (u(s) + v(s))^p ds \]
\[ \leq \varepsilon_3 M_1 \|(u, v)\|_Y^p \leq \frac{1}{4} \|(u, v)\|_Y, \]
\[ Q_2(u, v)(t) \leq \int_0^1 f_2(s) \varepsilon_4 (u(s) + v(s))^q ds \]
\[ \leq \varepsilon_4 M_2 \|(u, v)\|_Y^q \leq \frac{1}{4} \|(u, v)\|_Y. \]

These imply that
\[ \|Q_1(v, u)\| \leq \frac{1}{4} \|(u, v)\|_Y, \quad \|Q_2(u, v)\| \leq \frac{1}{4} \|(u, v)\|_Y, \]
\[ \|Q(u, v)\|_Y = \|Q_1(u, v)\| + \|Q_2(u, v)\| \leq \frac{1}{4} \|(u, v)\|_Y, \ \forall \ (u, v) \in \partial B_{r_1} \cap P. \]

From (Amann, 1976), we conclude that the fixed point index of operator \( Q \) over \( B_{r_1} \cap P \) with respect to \( P \) is
\[ i(Q, B_{r_1} \cap P, P) = 1. \]  

(13)

Combining (11) and (13) we obtain
\[ i(Q,(B_{r_1} \setminus \overline{B}_{r_1}) \cap P, P) = i(Q,B_{r_1} \cap P, P) - i(Q, B_{r_1} \cap P, P) = -1. \]

We deduce that \( Q \) has at least one fixed point \((u, v) \in (B_{r_1} \setminus \overline{B}_{r_1}) \cap P, \)
that is $r_1 < \|(u, v)\|_Y < R_1$ or $r_1 < \|u\| + \|v\| < R_1$. By Lemma 4, we obtain that $u(t) > 0$ for all $t \in (0, 1)$ or $v(t) > 0$ for all $t \in (0, 1)$. The proof is completed.

**Theorem 5.** Assume that (H1) and (H2) hold. If the functions $f$ and $g$ also satisfy the conditions

(H5) \[ f^*_0 = \limsup_{u, v \to 0} \sup_{t \in [0, 1]} \frac{f(t, u, v)}{u + v} = 0 \quad \text{and} \quad g^*_0 = \limsup_{u, v \to 0} \sup_{t \in [0, 1]} \frac{g(t, u, v)}{u + v} = 0; \]

(H6) There exist $\epsilon \in (0, 1)$, $\hat{\beta} \in (0, 1]$ and $\hat{q} \in (0, 1]$ such that

\[ f^*_0 = \limsup_{u, v \to 0} \inf_{t \in [c, 1]} \frac{f(t, u, v)}{(u + v)\hat{q}} = \infty \quad \text{and} \quad g^*_0 = \limsup_{u, v \to 0} \inf_{t \in [c, 1]} \frac{g(t, u, v)}{(u + v)\hat{q}} = \infty, \]

then problem (S)-(BC) has at least one positive solution $(u(t), v(t))$, $t \in [0, 1]$.

**Proof.** From the assumption (H5), we deduce that for $\epsilon_5 = \frac{1}{4M_1} > 0$ and $\epsilon_6 = \frac{1}{4M_2} > 0$, there exist $C_6 > 0$, $C_7 > 0$ such that

\[ f(t, u, v) \leq \epsilon_5(u + v) + C_6, \quad g(t, u, v) \leq \epsilon_5(u + v) + C_7, \quad \forall (t, u, v) \in [0, 1] \times [0, \infty) \times [0, \infty). \]

Hence for $(u, v) \in P$, by using (14), we obtain

\[ \begin{align*}
Q_1(u, v)(t) &\leq \int_0^1 J_1(s)(\epsilon_5(u(s) + v(s)) + C_6)ds \\
&\leq \epsilon_5(\|u\| + \|v\|) \int_0^1 J_1(s)ds + C_6 \int_0^1 J_1(s)ds \\
&= \epsilon_5(\|u\|_Y M_1 + C_6 M_1) + \frac{1}{4}\|u\|_Y + C_6 M_1, \quad \forall t \in [0, 1], \\
Q_2(u, v)(t) &\leq \int_0^1 J_2(s)(\epsilon_5(u(s) + v(s)) + C_7)ds \\
&\leq \epsilon_6(\|u\| + \|v\|) \int_0^1 J_2(s)ds + C_7 \int_0^1 J_2(s)ds \\
&= \epsilon_6(\|u\|_Y M_2 + C_7 M_2) + \frac{1}{4}\|u\|_Y + C_7 M_2, \quad \forall t \in [0, 1],
\end{align*} \]

and so

\[ \|Q(u, v)\|_Y = \|Q_1(u, v)\| + \|Q_2(u, v)\| \leq \frac{3}{4}\|u\|_Y + (C_6 M_1 + C_7 M_2) = \frac{3}{4}\|u\|_Y + C_8, \quad C_8 = C_6 M_1 + C_7 M_2. \]

Then there exists a sufficiently large $R_2 \geq \max\{4C_8, 1\}$ such that

\[ \|Q(u, v)\|_Y \leq \frac{3}{4}\|u\|_Y, \quad \forall (u, v) \in P, \|u\|_Y \geq R_2. \]

Hence \[\|Q(u, v)\|_Y < \|(u, v)\|_Y \text{ for all } (u, v) \in \partial B_{R_2} \cap P \text{ and from} \]

(Amman, 1976) we have

\[ \frac{1}{Q(B_{R_2} \cap P, P)} = 1. \]

On the other hand, from (H6) we deduce that for $\epsilon_7 = \frac{1}{4^m_{a-1} m_1} > 0$,

\[ \epsilon_8 = \frac{1}{c_{\epsilon_6}^{\frac{1}{m_2}}} > 0, \text{ there exists } r_2 \in (0, 1) \text{ such that} \]

\[ \frac{1}{c_{\epsilon_6}^{\frac{1}{m_2}}} > 0, \text{ there exists } r_2 \in (0, 1) \text{ such that} \]

\[ \frac{1}{c_{\epsilon_6}^{\frac{1}{m_2}}} > 0, \text{ there exists } r_2 \in (0, 1) \text{ such that} \]
\[
\begin{align*}
  \forall t \in [c, 1], \quad u, v \geq 0, \quad u + v \leq r_2.
  
  f(t, u, v) &\geq \epsilon_7 (u + v)^\beta, \quad g(t, u, v) \geq \epsilon_8 (u + v)^\gamma, \\
  Q_1(u, v)(t) &\geq \int_c^1 e_7 G_1(t, s) f(s, u(s), v(s)) ds \\
  &\geq e_7 \int_c^1 G_1(t, s)(u(s) + v(s))^\beta ds \\
  &\geq e_7 \int_c^1 G_1(t, s)(u(s) + v(s)) ds =: L_1(u, v)(t), \quad \forall t \in [0, 1], \\
  Q_2(u, v)(t) &\geq \int_c^1 G_2(t, s) g(s, u(s), v(s)) ds \\
  &\geq e_8 \int_c^1 G_2(t, s)(u(s) + v(s))^\gamma ds \\
  &\geq e_8 \int_c^1 G_2(t, s)(u(s) + v(s)) ds =: L_2(u, v)(t), \quad \forall t \in [0, 1].
\end{align*}
\]

Hence
\[
Q(u, v) \geq L(u, v), \quad \forall (u, v) \in \partial B_{r_2} \cap P,
\]
where the linear operator \( L: P \to P \) is defined by \( L(u, v) = (L_1(u, v), L_2(u, v)) \).

For \( \tilde{u}_0(t) = \int_c^1 G_1(t, s) ds \), \( \tilde{v}_0(t) = \int_c^1 G_2(t, s) ds \), \( \forall t \in [0, 1] \),
we have \( L(\tilde{u}_0, \tilde{v}_0) = (L_1(\tilde{u}_0, \tilde{v}_0), L_2(\tilde{u}_0, \tilde{v}_0)) \) with
\[
\begin{align*}
  L_1(\tilde{u}_0, \tilde{v}_0)(t) &= e_7 \int_c^1 G_1(t, s) \left( \int_c^1 G_1(s, \tau) d\tau + \int_c^1 G_2(s, \tau) d\tau \right) ds \\
  &\geq e_7 \int_c^1 G_1(t, s) \left( \int_c^1 G_1(t, s) ds \right) \\
  &= e_7 a_1 \int_c^1 G_1(t, s) ds = \int_c^1 G_1(t, s) ds = \tilde{u}_0(t), \quad \forall t \in [0, 1], \\
  L_2(\tilde{u}_0, \tilde{v}_0)(t) &= e_8 \int_c^1 G_2(t, s) \left( \int_c^1 G_1(s, \tau) d\tau + \int_c^1 G_2(s, \tau) d\tau \right) ds \\
  &\geq e_8 \int_c^1 G_2(t, s) \left( \int_c^1 G_2(s, \tau) d\tau \right) \\
  &= e_8 \int_c^1 G_2(t, s) ds = \tilde{v}_0(t), \quad \forall t \in [0, 1].
\end{align*}
\]
So
\[
L(\tilde{u}_0, \tilde{v}_0) \geq (\tilde{u}_0, \tilde{v}_0).
\]

We may suppose that \( Q \) has no fixed point on \( \partial B_{r_2} \cap P \) (otherwise the proof is finished). From (17), (18) and (Zhou and Xu, 2006, Lemma 3), we conclude that
\[
i(Q, B_{r_2} \cap P, P) = 0.
\]

Therefore, from (15) and (19), we have
\[
i(Q, B_{r_2} \setminus \tilde{B}_{r_2} \cap P, P) = i(Q, B_{r_2} \cap P, P) - i(Q, B_{r_2} \cap P, P) = 1.
\]

Then \( Q \) has at least one fixed point in \( (B_{r_2} \setminus \tilde{B}_{r_2}) \cap P \), that is \( r_2 < \|(u, v)\|_P < R_2 \). Thus problem (S)-(BC) has at least one positive solution \( (u, v) \in P \). This completes the proof.  \( \blacksquare \)
Theorem 6. Assume that (H1), (H2), (H4) and (H6) hold. If the functions \( f \) and \( g \) also satisfy the condition (H7) For each \( t \in [0,1] \), \( f(t,u,v) \) and \( g(t,u,v) \) are nondecreasing with respect to \( u \) and \( v \), and there exists a constant \( N_0 > 0 \) such that
\[
f(t,N_0,N_0) < \frac{N_0}{2m_0}, \quad g(t,N_0,N_0) < \frac{N_0}{2m_0}, \quad \forall \ t \in [0,1],
\]
where \( m_0 = \max\{M_i, \ i = 1,2\} \), \( M_i = \int_0^1 J_i(s)ds, \ i = 1,2 \),
then problem (S)-(BC) has at least two positive solutions \((u_1(t),v_1(t)), (u_2(t),v_2(t))\), \( t \in [0,1] \).

Proof. By using (H7), for any \((u,v) \in \partial B_{N_0} \cap P\), we obtain
\[
Q_1(u,v)(t) \leq \int_0^1 G_1(t,s) f(s,N_0,N_0) ds \leq \int_0^1 J_1(s) f(s,N_0,N_0) ds < \frac{N_0}{2m_0} \int_0^1 J_1(s) ds = \frac{N_0M_1}{2m_0} \leq \frac{N_0}{2}, \quad \forall \ t \in [0,1],
\]
\[
Q_2(u,v)(t) \leq \int_0^1 G_2(t,s) g(s,N_0,N_0) ds \leq \int_0^1 J_2(s) g(s,N_0,N_0) ds < \frac{N_0}{2m_0} \int_0^1 J_2(s) ds = \frac{N_0M_2}{2m_0} \leq \frac{N_0}{2}, \quad \forall \ t \in [0,1].
\]
Then we deduce
\[
\|Q(u,v)\|_Y = \|Q_1(u,v)\| + \|Q_2(u,v)\| < N_0, \quad \forall \ (u,v) \in \partial B_{N_0} \cap P.
\]
By (Amann, 1976) we conclude that
\[
i(Q,B_{N_0} \cap P, P) = 1. \tag{20}
\]
On the other hand, from (H4) and (H6) and the proofs of Theorem 4 and Theorem 5, we know that there exists a sufficiently large \( R_1 > N_0 \) and a sufficiently small \( r_2 \in (0,N_0) \) such that
\[
i(Q,B_{R_1} \cap P, P) = 0, \quad i(Q,B_{r_2} \cap P, P) = 0, \tag{21}
\]
From the relations (20) and (21), we obtain
\[
i(Q,(B_{R_1} \setminus \overline{B}_{N_0}) \cap P, P) = i(Q,B_{R_1} \cap P, P) - i(Q,B_{N_0} \cap P, P) = -1,
\]
\[
i(Q,(B_{N_0} \setminus \overline{B}_{r_2}) \cap P, P) = i(Q,B_{N_0} \cap P, P) - i(Q,B_{r_2} \cap P, P) = 1.
\]
Then \( Q \) has at least one fixed point \((u_1,v_1) \in (B_{R_1} \setminus \overline{B}_{N_0}) \cap P \) and has at least one fixed point \((u_2,v_2) \in (B_{N_0} \setminus \overline{B}_{r_2}) \cap P \). If in Theorem 5, the operator \( Q \) has at least one fixed point on \( \partial B_{r_2} \cap P \), then by using the first relation from formula above, we deduce that \( Q \) has at least one fixed point \((u_1,v_1) \in (B_{R_1} \setminus \overline{B}_{N_0}) \cap P \) and has at least one fixed point \((u_2,v_2) \in \partial B_{r_2} \cap P \). Therefore, problem (S)-(BC) has two distinct positive solutions \((u_1,v_1), (u_2,v_2)\). The proof is completed. \( \Box \)
REFERENCES


**EXISTENȚA SOLUȚIILOR POZITIVE PENTRU O PROBLEMĂ LA LIMITĂ FRAȚIONARĂ**

(Rezumat)

Studiem existența și multiplicitatea soluțiilor pozitive pentru sistemul de ecuații diferențiale fracționare Riemann-Liouville (S) cu neliniarități nenegative și nesingulare, cu condițiile la limită (BC) cu mai multe puncte care conțin derivate fracționare.