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HEUN SOLUTIONS' CLASS FOR AN EXTENSION OF KOMPANEETS KINETIC EQUATION

BY

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Abstract. An extended Kompaneets equation has been theoretically investigated. Closed form stationary solutions have been derived in terms of Heun confluent functions. These functions are governing the distribution of scattered photons.

Keywords: Heun confluent functions; extended Kompaneets equation; relativistic corrections; photon' spectrum; Compton scattering.

1. Introduction

Intensive studies over the interaction between radiation and electrons through the Compton scattering processes reveal a rich and very exciting evolutions of both the photon spectrum and the involved electrons. One may note that the comptonization process seen through a multiple scattering scenario is encountered at the level of relativistic corrections to the Sunyaev-Zeldovich effect (Taylor and Wright, 1989) which are revealing complex mathematical representation for the spectral distortion. When it comes to analyse mathematically the spectrum of the distorted radiation field it proves that one

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should deal with forms of Kompaneets equation, the most exciting for exploration being the corrective ones for the sake of relativistic effects.

In the last decades, with the help of SuZIE and MITO telescopes, a series of 16 clusters were observed at the level of higher frequencies (Rephaeli *et al*., 2005), where a significant intensity spectral change as a result of photon scattering by thermal electrons is to be recorded. Exact expressions for the relativistically photon intensity distortion are indispensible, most ardently when it comes to compute precise values of cluster and cosmological parameters.

In this study, we are mainly concerned in the correlation of the Kompaneets equation' extensions and its solutions in terms of Heun type ones. Nowadays, this topic being a fervent one, extensions on the Kompaneets equation to the relativistic regime having an outstanding role permitting, for instance, a more accurate determination of spectral quantities in higher energetic regimes.

2. Spectral Distribution of Photons Dictated by Heun Confluent Functions

The dimensional nonlinear Kompaneets equation (Kompaneets, 1957), also known as the photon diffusion equation, has the above mathematical formulation,

$$\frac{\partial n}{\partial t} = \frac{1}{x^2}\frac{\partial}{\partial x}\left[x^4\left(\alpha n_x + \beta n + \gamma n^2\right)\right], \tag{1}$$

where $\alpha > 0$, $\beta \geq 0$ and $\gamma > 0$ define some arbitrary constants. The change of distribution function $n(x,t)$ is treated as a diffusion of photon-gas in the 'frequency space' along the frequency axis $x \equiv h\nu / kT_e$.

Eq. (1) models the Compton scattering type interaction between a low-energy homogeneous photon gas and a rarefied electron gas. A more physical configuration proposed also by Kompaneets for the equation of radiative transfer, applicable in case of Comptonization-hardening for the energetic severe condition $h\bar{\nu} \langle\langle kT_e \langle\langle m_e c^2$, is

$$\frac{\partial n}{\partial t} = \frac{kT_e}{m_e c^2} N_e \sigma_T c \frac{1}{x^2}\frac{\partial}{\partial x}\left[x^4\left(\frac{\partial n}{\partial x} + n + n^2\right)\right]. \tag{2}$$

It appears that Eq. (2) is no longer adequate within the frame of hard X-ray astronomy where the condition $h\bar{\nu} \rangle\rangle kT_e$ is often valid. In this case, using it would imply the presence a significant error in the final results.

A significant improvement of Eq. (2), under the much looser condition $kT_e \langle\langle m_e c^2$ and $h\nu \langle\langle m_e c^2$, has been deduced by D.B. Liu *et al*. in (Liu *et al*., 2004) by extending the Kompaneets equation

$$\frac{\partial n}{\partial t} = \frac{kT_e}{m_e c^2} N_e \sigma_T c \frac{1}{x^2} \frac{\partial}{\partial x} \left\{ x^4 \left(1 + \frac{7}{10} \frac{kT_e}{m_e c^2} x^2 \right) \left[\frac{\partial n}{\partial x} + n(1+n) \right] \right\}. \quad (3)$$

In the study of Sazonov and Sunyaev (Sazonov and Sunyaev, 2000), various elaborated generalized forms of kinetic equation which consider a density of correction terms responsible, for instance, for quantum effects, induced scattering or Doppler effect, have to be found. Relativistic corrections present within these versions of Kompaneets equation may play a significant role in problems such as the formation of shock waves in the photon spectrum during the phenomenon of Bose condensation of photons (Zeldovich and Sunyaev, 1972) or plasma heating (Levich and Sunyaev, 1971).

Within our study, following an analytical approach, we will focus on finding closed form solutions for the following extension of the Kompaneets equation:

$$\frac{\partial n}{\partial t} = \frac{1}{x^2} \frac{\partial}{\partial x} \left\{ x^4 \left(1 + a x^2 \right) \left[b x \frac{\partial n}{\partial x} + n(1+n) \right] \right\} \quad (4)$$

We mention that our class of extended Kompaneets equation, namely Eq. (4), is similar to the one proposed by Liu and coworkers, with the single distinction, the one that the Doppler term is being attended by a pre-factor.

Within the hypothesis of stationary, this equation is reduced to the Riccati type form

$$b x \frac{\partial n}{\partial x} + n + n^2 = \frac{Q}{x^4 \left(1 + a x^2 \right)} \quad (5)$$

with Q, as we find in (Dubinov, 2009), defining a constant whose physical meaning is of the photon flux in the implied frequency domain. The Riccati types equations are frequently encountered within the Kompaneets studies where these are discussed in detail.

In order to solve Riccati form (5), firstly we will invoke the change of variable $y = (1/b) \ln x$ which leads to the new representation

$$\frac{\partial n}{\partial y} + n + n^2 = \frac{Q}{e^{4by} \left(1 + a e^{2by} \right)}. \quad (6)$$

At this step, the change of function given by

$$n = \frac{\Psi'}{\Psi} \tag{7}$$

provides us with a second order differential equation for the function Ψ:

$$\Psi'' + \Psi' - \frac{Q}{e^{4by}\left(1 + ae^{2by}\right)}\Psi = 0 \tag{8}$$

A new change of function given by $\Psi = \exp\left[-\frac{\sqrt{Q}e^{-2by} + 2by}{2b}\right] \cdot u(y)$ leads to the following differential equation for the unknown function $u(y)$:

$$\begin{aligned} &\frac{d^2u}{dy^2} + \left(2\sqrt{Q}e^{-2by} + 1\right)\frac{du}{dy} + \\ &+\left(-2b\sqrt{Q}e^{-2by} + e^{-4by}Q + \sqrt{Q}e^{-2by} - \frac{Q}{e^{4by}\left(1 + ae^{2by}\right)}\right)u = 0 \end{aligned}, \tag{9}$$

the solution of this equation being given by

$$u(y) = e^{-y} HeunC\left[\frac{a\sqrt{Q}}{b}, \frac{1}{2b}, -1, -\frac{a^2Q}{4b^2}, \frac{1}{2}, -\frac{e^{-2by}}{a}\right] \tag{10}$$

so that we can deduce that

$$\Psi = x^{-1/b} \cdot \exp\left[-\frac{\sqrt{Q}}{2bx^2}\right] \cdot HeunC\left[\frac{a\sqrt{Q}}{b}, \frac{1}{2b}, -1, -\frac{a^2Q}{4b^2}, \frac{1}{2}, -\frac{1}{ax^2}\right] \tag{11}$$

This expression allows us to determine the intricate photon distribution function defined in (7) in terms of the *HeunC* functions,

$$n = \frac{\sqrt{Q} - x^2}{bx^3} + \frac{HeunC'}{HeunC}. \tag{12}$$

To be noted that the spectrum of the scattered photons is intimately dependent on the photon flux Q in the respective frequency region.

For practical calculations, it is more elegant and transparent to recall the polynomial representation of the Heun confluent function. Heun function, $HeunC[\alpha, \beta, \gamma, \delta, \eta, z]$, admits a polynomial representation when the series expansion for it truncates, so that in this situation the function degenerates into a polynom. In our case, by invoking the algebraic constriction (necessary, but not sufficient) for this to happen, namely

$$\delta = -\left(n + \frac{\gamma + \beta + 2}{2}\right)\alpha , \tag{13}$$

the identified parameters in (12) makes us obtain the relation

$$\frac{a\sqrt{Q}}{4b} = n + \frac{2b+1}{4b} \tag{14}$$

which is equivalent with the following parametric constriction

$$a\sqrt{Q} = 4nb + 2b + 1 . \tag{15}$$

To be noted that n is a positive integer defining the polynom' degree.

For more insights into polynomial representations, we recommend the study in (Fiziev, 2010) where it is to be found an innovative derivation of confluent Heun's polynomials.

As Heun functions are mathematically difficult to operate with, due to the singularities or to their problematic act of derivation a series, their polynomial forms are very useful. These allow a compactified and a more tractable representation.

3. Conclusions

Within an analytical approach, closed form solutions for a relativistically corrected Kompaneets equation have been determined. The spectrum of the scatterred photons proved to be non-trivially determined by the Heun confluent functions and intimately connected with the photon flux in the respective frequency domain. The polynomial Heun solution has been discussed in terms of parametric constriction.

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REFERENCES

Dubinov A.E., *Exact Stationary Solution of the Kompaneets Kinetic Equation*, Tech. Phys. Lett., **35**, 260-262 (2009).

Fiziev P.P., *Novel Relations and New Properties of Confluent Heun's Functions and their Derivatives of Arbitrary Order*, J. Phys. A: Math. Theor., **43**, 035203-035211 (2010).

Kompaneets A.S., *The Establishment of Thermal Equilibrium between Quanta and Electrons*, Sov. Phys., JETP, **4**, 730-737 (1957).

Levich E.V., Sunyaev R.A., *Heating of Gas Near Quasars, Seyfert-Galaxy Nuclei, and Pulsars by Low-Frequency Radiation*, Soviet Astron. Lett., **15**, 363-370 (1971).

Liu D.–B., Chen L., Ling J.J., You J.H., Hua X.M., *Evolution of X-Ray Spectrum in Down-Comptonization Process. A Comparison of the Extended Kompaneets Equation with Monte Carlo Simulation and the Ross-McCray Equation*, A&A, **417**, 381-389 (2004).

Rephaeli Y., Sadeh S., Shimon M., *The Sunyaev-Zeldovich Effect*, Riv. Nuovo Cim., **29N12**, 1-18 (2006).

Sazonov S.Y., Sunyaev R.A., *The Profile of a Narrow Line after Single Scattering by Maxwellian Electrons: Relativistic Corrections to the Kernel of the Integral Kinetic Equation*, Astrophys. J., **543**, 28-55 (2000).

Taylor G.B., Wright E.L., *Effects of a Hot Intergalactic Medium*, ApJ, **339**, 619-628 (1989).

Zeldovich Ya.B., Sunyaev R.A., *Shock Wave Structure in the Radiation Spectrum During Bose Condensation of Photons*, Soviet Phys.-JETP, **35**, 81-85 (1972).

CLASA DE SOLUȚII HEUN PENTRU O EXTENSIE A ECUAȚIEI CINETICE KOMPANEETS

(Rezumat)

O formă extinsă a ecuației Kompaneets a fost investigată teoretic. Au fost determinate soluții staționare în formă închisă, exprimate prin funcții Heun confluente. Aceste funcții guvernează distribuția fotonilor împrăștiați. Soluția polinomială Heun a fost analizată în termenii constrângerii parametrice.