BULETINUL INSTITUTULUI POLITEHNIC DIN IAŞI
Publicat de
Universitatea Tehnică „Gheorghe Asachi” din Iaşi
Volumul 64 (68), Numărul 4, 2018
Secţia
MATEMATICĂ. MECANICĂ TEORETICĂ. FIZICĂ

HEUN SOLUTIONS FOR CHARGED BOSONS IN AXIALLY SYMMETRIC MAGNETIC FIELDS

BY

NARCISA MURARIU[*] and MARINA AURA DARIESCU

“Alexandru Ioan Cuza” University of Iaşi, Romania,
Faculty of Physics

Received: November 16, 2018
Accepted for publication: December 14, 2018

Abstract. The present paper focuses on an analytical analysis which describes the evolution of the relativistic charged bosons in magnetar’s crust. Starting with the Klein-Gordon equation, the Heun Biconfluent functions are obtained for a strong static magnetic induction orthogonal to a radial electric field.

Keywords: Heun equation; Klein-Gordon equation; magnetars; static magnetic fields; Riemann Zeta function.

1. Introduction

With a density up to an order of magnitude higher than the one of the nuclear matter, neutron stars are seen as laboratories where different subatomic particle processes are competing with each other (Weber, 1999).

By studying the temporal variations in the incoming flux, the satellites are detecting the arrival time (microseconds) of each photon that hits the detector and plot the accumulated count rate as a function of time, $c(t)$. This is than transformed to the frequency domain, using the Fourier analysis, and one gets the normalized power density spectrum.

---

[*]Corresponding author; *e-mail*: murariu.narcisa@gmail.com

In the case of radio pulsars (about 1600 are known today), by timing the arrivals of radio pulses, the period of rotation and its derivative can be estimated. Assuming that the pulsar spin down is due only to magnetic dipole radiation, one can easily identify the rotation energy loss

$$\frac{dE}{dt} = -4\pi^2 I \frac{\dot{P}}{P^3},$$

P being the spin period and $\dot{P}$ its derivative, with the total time-average dipole radiated power defined as

$$w = \frac{\omega^4}{6\pi\varepsilon_0 c^3} |\vec{p}|^2,$$

and he gets

$$w = \frac{32\pi^4}{3c^3} \frac{B^2 R^6}{P^4}, \text{ for } 4\pi\varepsilon_0 = 1.$$

Thus, one comes to the following relation for the surface magnetic field induction

$$B^2 = \frac{3c^3}{8\pi^2} \frac{I}{R^6} P\dot{P},$$

where P is in units of seconds.

In 1992, Duncan and Thompson introduced the notion of magnetars (Duncan and Thompson, 1992), for almost non-rotating neutron stars, whose magnetic field strength was estimated to be about 10^2 to 10^3 larger than the one of a neutron star. Now, by the name magnetars, we call eight AXPs (anomalous X-ray pulsars) and four SGRs (Soft gamma repeaters), which share several characteristics (Thompson and Beloborodov, 2005):

- persistent X-ray luminosities in the range $10^{34} - 10^{36}\,\text{erg/s}$;
- spin periods between 2 and 12 s;
- characteristic ages

$$T \sim \frac{P}{2\dot{P}} \sim 10^3 - 10^5\,\text{yrs},$$

- the release of short $(\sim 0.1\text{s})$ and spectrally hard X-ray bursts;
- and a strong magnetic field, $B \sim 10^{14} - 10^{15}\,\text{G}$.

Presently, among the almost 1800 spindown-powered radio pulsars are known, with periods from about 1.5 ms to 8 s and an average magnetic field of $\sim 10^{12}$ G, the magnetars are a small group, characterized by a very strong X-ray emission. This lead (when treating them as magnetic dipole radiators) to a magnetic field of about B~10^{14}–10^{15} G, larger than the critical induction at which the cyclotron energy of an electron equals the electron rest mass energy.

Up to now, magnetars are considered the only source of giant flares, which are the brightest cosmic events originating outside the solar system, in terms of the flux received at Earth.

In the last almost 20 years since Duncan and Thompson published their seminal work (Duncan and Thompson, 1992), the intense study on such astrophysical objects has led to many open questions, especially related to the configuration of the magnetic field inside and to their structure. Besides its birth, the mechanism of converting the strong magnetic fields into radiation has also been a main topic of investigations. In this respect, many ideas are coming from the so-called Suzaku experiment (Mitsuda *et al*., 2007), which has performed pointed observations on bright presently known magnetars.

In what it concerns the theoretical description of the magnetic field, most of the proposed models are dynamical, with the magnetic field evolving from the birth to the star's decay, through different processes (Goldreich and Reisenegger, 1992). However, soon after the crust forms, (around 100 s after collapse), the magnetic field is freezing, and they can be treated as stationary objects.

For many years, the equation derived by Goldreich and Reisenegger (1992) has been considered as a reliable tool for understanding the evolution of both isolated and accreting neutron stars, due to the Hall effect and Ohmic diffusion (Hoyos *et al*., 2008; Cumming *et al*., 2004),

$$\frac{\partial \vec{B}}{\partial t} = -\frac{1}{\sigma}\nabla\times\left[\nabla\times\vec{B}\right] + \nabla\times\left[-\frac{\vec{j}}{ne}\times\vec{B}\right].$$

For some characteristic depth of the crust, d, the Ohmic decay time is given by

$$\tau_{Ohm} = \sigma d^2 = \frac{d^2}{\rho} = 3\times10^5\left(\frac{d}{1\text{km}}\right)^2\left(\frac{\rho}{10^{-3}\text{cm}^2\text{s}^{-1}}\right)^2 \text{yr},$$

which is a typical value for the outer crust at temperature $T = 10^8\,\text{K}$ and the Hall time is

$$\tau_{Hall} = 4\pi\frac{ned^2}{cB} \sim 6\left(\frac{d}{1\text{km}}\right)^2\left(\frac{B}{10^{15}\text{G}}\right)^2 \text{yr},$$

where n is the electron number density.

Thus, for strong magnetic fields, the Hall time is several orders of magnitude faster than the Ohmic one. The Hall drift evolves the magnetic field into a new configuration of equal total (conserved) energy, being unable to cause decay by itself. However, by changing the field structure on Hall timescale, it may give rise to a turbulent cascade, enhancing the efficiency of Ohmic total energy decay, as Goldreich and Reisenegger suggested, in 1992.

The energy is transferred from large to small scales, with an energy spectrum $E_k \sim \frac{1}{k^2}$, and a dissipative cutoff occurring at $k \sim B_0$.

Following Goldreich and Reisenegger's seminal work, numerous authors have studied this phenomenon via theoretical and numerical methods, in both spherical-shells (Hollerbach and Rudiger, 2004) and Cartesian box geometries (Biskamp *et al.*, 1999).

A neutron star is made of atmosphere and four main internal regions: the outer and inner crust and the outer and inner core (containing up to 99% of the star's mass).

While the outer core consists of a soup of nucleons, electrons and muons, the matter in the inner core may be compressed to densities that are up to an order of magnitude greater than the density of ordinary atomic nuclei, depending on star mass and rotational frequency. This extreme compression provides a high-pressure environment in which exotic subatomic particles may be formed.

Even though there is a large activity in the field, which has raised some debatable issues, until now studies on the contribution of meson resonances in stars with dying magnetic fields have not been done to a large extent. One can expect that, for a neutron star with a wide range of densities, from the density of iron nucleus at the surface to several times the normal nuclear matter density in the core, the existence of boson condensates has an important influence on the star properties (Pal *et al.*, 2000). Based on results obtained by Kaplan and Nelson (1986), who have studied the negatively charged antikaon condensation in dense baryonic matter formed in heavy-ion collisions, the kaons are seen as best candidates for matter inside superdense astrophysical objects, besides nucleons and leptons.

One may conclude by saying that there is no doubt that scalar fields, if exist, are leading to fingerprints in the observable stellar quantities that can be measured with the advanced technology of observational radio and X-ray astronomy.

2. Klein-Gordon Equation for Bosons in Magnetar's Crust

For describing the relativistic complex charged bosons of mass m_0, evolving in the magnetar's crust with a static magnetic field orthogonal to an electric field, we start with the U(1)-gauge invariant Lagrange density

$$L = \eta^{ij} \left(D_i \Psi\right)^* D_j \Psi + \frac{m_0^2 c^2}{\hbar^2} \Psi^* \Psi, \tag{1}$$

where D_i stands for the U(1)-gauge covariant derivative, with $i=\overline{1,4}$,

$$D_i\Psi = \Psi_{,i} - \frac{i}{\hbar}qA_i\Psi,$$
$$D_i\Psi^* = \Psi^*_{,i} + \frac{i}{\hbar}qA_i\Psi^*, \tag{2}$$

and we have inserted h and c for a better comparison with data.

We use the general form of the gauge potential

$$A_x = -\frac{B_0}{2}y - \frac{y\phi}{2\pi\left(x^2+y^2\right)},$$
$$A_y = \frac{B_0}{2}x + \frac{x\phi}{2\pi\left(x^2+y^2\right)},$$
$$A_z = \frac{\eta}{q}, \tag{3}$$
$$A^4 = -A_4 = \frac{V\left(\vec{x}\right)}{c},$$

which satisfy the condition $\partial_i A^i = 0$

The expressions (3) are corresponding to a constant magnetic field along the Oz-direction,

$$B_z = F_{12} = \partial_x A_y - \partial_y A_x \equiv B_0,$$

and a static electric field $\vec{E} = -\nabla V$. By employing the usual procedure, we come to the corresponding Euler-Lagrange equation

$$\eta^{ij}D_iD_j\Psi - \frac{m_0^2c^2}{\hbar^2}\Psi = 0, \tag{4}$$

whose explicit form is

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]\Psi + iq\frac{B_0}{\hbar}\left[y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right]\Psi - \frac{q^2B_0^2}{4\hbar^2}\left(x^2+y^2\right)\Psi +$$
$$+\frac{iq\phi}{\pi\hbar(x^2+y^2)}\left[y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right]\Psi - \frac{q^2B_0\phi}{2\pi\hbar^2}\Psi - \frac{q^2\phi^2}{4\pi^2\hbar^2\left(x^2+y^2\right)}\Psi +$$
$$+\frac{\partial^2\psi}{\partial z^2} - 2i\frac{\eta}{\hbar}\frac{\partial\psi}{\partial z} - \frac{\eta^2}{\hbar^2}\Psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - 2i\frac{qV}{\hbar c^2}\frac{\partial\psi}{\partial t} + \frac{q^2V^2}{\hbar^2c^2}\Psi - \tag{5}$$
$$-\frac{m_0^2c^2}{\hbar^2}\Psi = 0$$

The above formula suggests us to switch to cylindrical coordinates:

$$\{x = \rho\cos\varphi, y = \rho\sin\varphi, z\} ,$$

where

$$x + iy = \rho e^{i\varphi}, x - iy = \rho e^{-i\varphi}$$

and

$$\partial_x + i\partial_y = e^{i\varphi}\left(\partial_\rho + \frac{i}{\rho}\partial_\varphi\right), \partial_x - i\partial_y = e^{-i\varphi}\left(\partial_\rho - \frac{i}{\rho}\partial_\varphi\right)$$

With the standard variable's separation

$$\Psi = \chi(\rho)e^{im\varphi}\exp\left[\frac{i}{\hbar}(pz - \omega t)\right], \tag{6}$$

the Eq. (5) leads to the following differential equation for the ρ depending part,

$$\begin{aligned}&\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\chi}{d\rho}\right) - \frac{1}{\rho^2}\left(m - \frac{q\phi}{2\pi\hbar}\right)^2\chi + \frac{qB_0}{\hbar}\left(m - \frac{q\phi}{2\pi\hbar}\right)\chi - \\ &- \frac{q^2B_0^2}{4\hbar^2}\rho^2\chi + \frac{1}{\hbar^2}\left[\frac{(\omega - qV)^2}{c^2} - k^2 - m_0^2c^2\right]\chi = 0,\end{aligned} \tag{7}$$

where we have introduced the notation $k \equiv p - \eta = const$.

One may notice that, with the new quantum number

$$l = m - \frac{q\phi}{2\pi\hbar}, \tag{8}$$

and the change of function

$$\chi(\rho) = \rho^l \exp\left[-\frac{qB_0}{4\hbar}\rho^2\right]w(\rho), \tag{9}$$

the Eq. (7) gets the simpler expression

$$\frac{d^2w}{d\rho^2} + \left[\frac{2l+1}{\rho} - \frac{qB_0}{\hbar}\rho\right]\frac{dw}{d\rho} + \frac{1}{\hbar^2}\left[\frac{(\omega - qV)^2}{c^2} - k^2 - m_0^2c^2 - qB_0\hbar\right]w = 0 \tag{10}$$

whose integration procedure strongly dependents on the explicit form of the electric potential $V(\rho)$.

3. The Particular Case of the Static Magnetic Field

The case of the zero or constant potential, $V(\rho)\equiv V_0$ is completely analytically treatable, since the new variable

$$s=\frac{qB_0}{2\hbar}\rho^2 \tag{11}$$

brings the differential Eq. (10) to the form

$$s\frac{d^2w}{ds^2}-[s-l-1]\frac{dw}{ds}+\frac{1}{2qB_0\hbar}\left[\frac{(\omega-qV_0)^2}{c^2}-k^2-m_0^2c^2-qB_0\hbar\right]w=0. \tag{12}$$

If one imposes the condition

$$\frac{(\omega-qV_0)^2}{c^2}-k^2-m_0^2c^2-qB_0\hbar=2nqB_0\hbar, \tag{13}$$

leading to the energy levels

$$\omega_n=qV_0\pm m_0c^2\sqrt{1+(2n+1)\frac{\hbar qB_0}{m_0^2c^2}+\frac{k^2}{m_0^2c^2}}, \tag{14}$$

the solutions to (12) are the general Laguerre polynomials

$$w(s)=C_{nl}e^{-s/2}s^{l/2}L_n^{(l)}(s)$$

satisfying the normalization condition

$$\int_0^\infty e^{-s}s^l\left[L_n^{(l)}(s)\right]^2ds=\frac{(n+l)!}{n!}. \tag{15}$$

Putting everything together, we come to the following solution for the field equation describing a charged boson evolving in constant magnetic and electric fields B_0 and V_0:

$$\Psi=C_{nl}\exp\left[\frac{i}{\hbar}(pz-\omega t)\right]e^{im\varphi}\rho^l\exp\left[-\frac{qB_0}{4\hbar}\rho^2\right]L_n^{(l)}\left(\frac{qB_0}{2\hbar}\rho^2\right), \tag{16}$$

with $l = m - \frac{q\phi}{h}$ and

$$C_{nl} = \sqrt{\frac{n!}{(l+n)!}}.$$

Let us focus on the second term in the r.h.s. of (14), which is expressing the quantum mechanics contribution. In the semi-relativistic limit and for a weak electric field, we define the rescaled Newtonian energy by $\varepsilon = \omega - m_0 c^2$. For $\varepsilon^2 \approx 0, \varepsilon V_0 \approx 0, V_0^2 \approx 0$, we end up with the following Landau-type energy levels:

$$\varepsilon_n = qV_0 + \frac{k^2}{2m_0} + \left(n + \frac{1}{2}\right)\hbar\omega_c, \tag{17}$$

where $\omega_c = q\frac{B_0}{m_0}$.

For $m \neq q\phi/h$, these are highly degenerated and can be filled by $n_B = \phi/\phi_0$ charged particles of flux quanta $\phi_0 = h/q$ per unit area. The ration between the number of electrons, N and n_B is called the filling factor.

$$\nu = \frac{N}{n_B}$$

and can be integer or fractional, as in the integer or fractional QHE. The experimentally observed filling factor $\nu = 1/2$ has led to the idea of a condensation of electrons into composite quasi particles with a filling factor equal to one. Thus, the fractional Hall effect for electrons can be related to the integer Hall effect, for these composite particles.

However, there are many open questions left, as for example the one on the impact of the spin of such pairs. In the case under consideration, we have noticed that the integer $n_B = q\phi/h$ is affecting the azimuthal number m, which is switched to $l = m - n_B$. Consequently, a change of the ϕ by n_B flux quanta moves a state described by $\Psi_{n,m}$ to $\Psi_{n,m-n_B}$ (up to a phase $e^{in_B\varphi}$).

The non-relativistic electromagnetic current density defined by

$$j_i = -\frac{\hbar}{2m_0}\left[iq\left(\Psi^*\partial_i\Psi - \Psi\partial_i\Psi^*\right) + \frac{2q^2}{\hbar}|\Psi|^2 A_i\right], \tag{18}$$

has the following components coupled to $A_\rho = 0, A_\varphi = B_0 \dfrac{\rho}{2}$ and $A_z = \dfrac{\eta}{q}$,

$$j_\rho = 0,$$
$$j_\varphi = \frac{q\hbar}{m_0} l|\Psi|^2 - \frac{q^2 B_0}{2m_0}\rho|\Psi|^2, \tag{19}$$
$$j_z = \frac{q}{m_0} k|\Psi|^2,$$

while the charge density is

$$Q = \frac{1}{c} j^4 = -\frac{1}{c} j_4 = q\frac{\omega - qV_0}{m_0 c^2}|\Psi|^2. \tag{20}$$

One may notice that the non-trivial current density j_φ generates an additional z-oriented magnetic field

$$H_z(r) = -\int_0^r j_\varphi(\rho)d\rho = \frac{q^2 B_0}{2m_0}\int_0^r |\Psi|^2 \rho d\rho - \frac{q\hbar}{m_0} l\int_0^r |\Psi|^2 \frac{d\rho}{\rho} \tag{21}$$

that vanishes for $r^2 \gg r_B^2$,

$$\int_0^\infty j_\varphi(\rho)d\rho = 0, \tag{22}$$

where $r_B \equiv \sqrt{\hbar/(qB_0)}$ is the magnetic length. Using the normalization condition (15), which, for the wave function (16) turns into:

$$\int_0^\infty |\Psi|^2 \rho d\rho = \frac{\hbar}{qB_0}\left(\frac{2\hbar}{qB_0}\right)^l \tag{23}$$

the axially generated magnetic field intensity (21) is given in terms of the cylinder radius and the magnetic induction as being:

$$H_z = -\frac{q\hbar}{2m_0}\left(r_B^2\right)^l \left(\frac{r^2}{r_B^2}\right)^l \exp\left(-\frac{r^2}{2r_B^2}\right)\left[C_1 + C_2\frac{r^2}{r_B^2} + \ldots + C_n\left(\frac{r^2}{r_B^2}\right)^{2n}\right] \tag{24}$$

with

$$C_n = \frac{C_{n+l}^l}{l!}$$

and $C_n < C_{n-1} ... < C_2 < C_1$.

Finally, let us focus on the charge density (20), which, in view of the relation (16), is given by

$$Q = q \sum_n \sum_m |\Psi|^2 \sqrt{1 + (2n+1) \frac{\hbar q B_0}{m_0^2 c^2}} \tag{25}$$

Up to the first contribution in $r^2/(2r_B^2)$, its integration is

$$\int Q d\tau = 2\pi L \int_0^r Q \rho d\rho$$

$$\int Q d\tau = q\pi L \left(2r_B^2\right)^{1-n_B} \left[1 - \exp\left(-\frac{r^2}{2r_B^2}\right)\right] \times \\ \times \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} \left(2r_B^2\right)^m \sqrt{1 + (2n+1) \frac{\hbar \omega_c}{m_0 c^2}} \tag{26}$$

being expressed in terms of Riemann Zeta-function and its generalization,

$$\zeta(s,a) = \sum_k \frac{1}{(k+a)^s},$$

$$\Phi(z,s,a) = \sum_k \frac{z^k}{(k+a)^s},$$

as:

$$\int Q d\tau = q\pi L \left(2r_B^2\right)^{1-n_b} \sqrt{2 \frac{m_0 c^2}{\hbar \omega_c}} \left[1 - \exp\left(-\frac{r^2}{2r_B^2}\right)\right] \mathrm{x} \\ \mathrm{x} \left\{ \zeta\left[-\frac{1}{2}, \frac{1}{2}\left(3 + \frac{m_0 c^2}{\hbar \omega_c}\right)\right] - \left(2r_B^2\right) \Phi\left[2r_B^2, -\frac{1}{2}, \frac{1}{2}\left(3 + \frac{m_0 c^2}{\hbar \omega_c}\right)\right] \right\} \tag{27}$$

depending on r, B_0 and on the number of flux quanta n_B .

4. The Linear Potential and the Heun Functions

Let us move now to a physically important case, namely the one of the linear potential

$$V(\rho) \equiv -V_0 \frac{\rho}{a},$$

which generates a constant radial electric intensity $E_p = V_0/a$.

One may use the change of function

$$\chi(\rho) = \rho^l \exp\left[-\frac{qB_0}{4\hbar}\rho^2 + \frac{2\omega V_0}{\hbar c^2 a B_0}\rho\right] w(\rho), \tag{28}$$

for which the Eq. (7) turns into

$$\begin{aligned} &\frac{d^2 w}{d\rho^2} + \left[\frac{2l+1}{\rho} - \frac{qB_0}{\hbar}\rho + \frac{4\omega V_0}{\hbar c^2 a B_0}\right]\frac{dw}{d\rho} + \\ &+\left\{\frac{1}{\hbar^2 c^2}\left[\omega^2 - k^2c^2 - m_0^2 c^4 - qB_0\hbar c^2\right] + \right. \\ &\left. +\frac{4\omega^2 V_0^2}{\hbar^2 c^4 a^2 B_0^2} + \frac{(2l+1)}{\rho}\frac{2\omega V_0}{\hbar c^2 a B_0}\right\} w = 0. \end{aligned} \tag{29}$$

In terms of the dimensionless variable

$$\zeta = \sqrt{\frac{qB_0}{2\hbar}}\rho,$$

the Eq. (29) becomes

$$\begin{aligned} &\frac{d^2 w}{d\zeta^2} + \left[\frac{2l+1}{\zeta} - 2\zeta + \frac{4\sqrt{2}\omega V_0}{\hbar c^2 a B_0}\sqrt{\frac{\hbar}{qB_0}}\right]\frac{dw}{d\zeta} + \\ &+\left\{\frac{2}{qB_0\hbar c^2}\left[\omega^2 - k^2c^2 - m_0^2 c^4 - qB_0\hbar c^2 + \frac{4\omega^2 V_0^2}{c^2 a^2 B_0^2}\right] + \right. \\ &\left. +\frac{(2l+1)}{\zeta}\frac{2\sqrt{2}\omega V_0}{\hbar c^2 a B_0}\sqrt{\frac{\hbar}{qB_0}}\right\} w = 0. \end{aligned} \tag{30}$$

This has the form of the so-called Heun Biconfluent equation (Heun, 1889)

$$\frac{d^2u}{d\zeta^2}+\frac{1}{\zeta}\left[-2\zeta^2-\beta\zeta+\alpha+1\right]\frac{du}{d\zeta}+ \\ +\left[\left(\gamma-\alpha-2\right)\zeta-\frac{\delta+\beta+\alpha\beta}{2}\right]\frac{u}{\zeta}=0, \tag{31}$$

whose parameters are:

$$\alpha=2l, \\ \beta=-\frac{4\sqrt{2}\omega V_0}{\hbar c^2B_0}\frac{r_B}{a}, \\ \gamma=2\left(l+1\right)+\frac{2}{qB_0\hbar c^2}\left[\omega^2-k^2c^2-m_0^2c^4-qB_0\hbar c^2+\frac{4\omega^2V_0^2}{c^2a^2B_0^2}\right], \\ \delta=0 \tag{32}$$

The necessary condition for a polynomial form of $u=HeunB\left[\alpha,\beta,\gamma,\delta,\zeta\right]$ being (Decarreau *et al*., 1978)

$$\gamma=2n+\alpha+2=2\left(n+l+1\right)$$

we get the following quantized energy levels

$$\omega_n^2=\left[k^2c^2+m_0^2c^4+\left(n+1\right)qB_0\hbar c^2\right]\left[1+\frac{4V_0^2}{c^2a^2B_0^2}\right]^{-1}. \tag{33}$$

By setting to zero the coefficient of ζ^2 in the series expansion of HeunB, for $\delta=0$ and $n=1$, we get the following equation for the parameter α,

$$\beta^2\alpha^2+4\alpha\left(\beta^2-2\right)+3\beta^2-8=0,$$

whose solutions are:

$$\alpha_1=-1,\alpha_2=\frac{8}{\beta^2}-3.$$

For $\alpha_1=-1$, *i.e.* $l=-1/2$, the solution (28) has the concrete expression

$$\chi(\zeta)=N\frac{1}{\sqrt{\zeta}}\exp\left[-\frac{\zeta^2}{2}-\frac{\beta}{2}\zeta\right]HeunB\left[-1,\beta,3,0,\zeta\right]$$

where the constant N comes from the normalization

$$\int_0^\infty\int_0^{2\pi}\chi_{n,l}^+\chi_{n,l}\rho d\rho d\varphi=\frac{2h}{qB_0}\int_0^\infty\left|\chi_{n,l}\right|^2\zeta d\zeta\equiv 1.$$

To first order in ζ, *i.e.* $\rho\ll\sqrt{2}r_B$, using the series expansion of the Heun function (Slavyanov and Lay, 2000)

$$HeunB\left[\alpha,\beta,\gamma,\delta,z\right]\approx 1+\frac{\delta+\beta\alpha+\beta}{2\alpha+2}z+\mathrm{O}\left(z^2\right)$$

namely

$$HeunB=1+\frac{\beta}{2}\zeta, \tag{34}$$

the wave function (6) reads

$$\Psi=Ne^{im\varphi}\exp\left[\frac{i}{\hbar}\left(pz-\omega t\right)\right]\frac{1}{\sqrt{\zeta}}\exp\left[-\frac{\zeta^2}{2}-\frac{\beta}{2}\zeta\right]\left[1+\frac{\beta}{2}\zeta\right], \tag{35}$$

with $m=n_B-\dfrac{1}{2}$ and

$$\beta\zeta=\frac{4\omega V_0}{\hbar c^2 B_0}\frac{\rho}{a}.$$

Obviously, for $V_0=0$ so that $\beta=\delta=0$ in (32) and $\alpha=1$, the solutions of (30) are

$$u\sim\frac{1}{\zeta}H_n\left(\zeta\right),$$

where H_n are the Hermit polynomials and,

$$\omega_n=c\sqrt{nqB_0\hbar+k^2+m_0^2c^2}$$

while the wave function (28) is expressed in terms of the Hermit associated function as

$$\chi \sim \frac{1}{\sqrt{\zeta}} \exp\left[-\frac{\zeta^2}{2}\right] H_n(\zeta) \ .$$

For arbitrary $\alpha = 2l$, the solution of (30) is the confluent hypergeometric functions

$$u \sim {}_1F_1\left[-\frac{n}{2}, l+1, \zeta^2\right]$$

5. Conclusions

In cylindric coordinates and for a constant magnetic field, the relativistic charged bosons in a magnetar's crust is described by the Laguerre polynomials. The wave function leads to concrete expressions for the conserved current density components. The flux quantization and the usual Landau type energy levels are obtained.

For the physical case of a linear radial potential, the more involved case of the Heun Biconfluent equation is discussed.

Acknowledgements. This work was supported by a grant of Ministery of Research and Innovation, CNCS - UEFISCDI, project number PN-III-P4-ID-PCE-2016-0131, within PNCDI III.

REFERENCES

Biskamp D. *et al.*, *Electron Magnetohydrodynamic Turbulence*, Physics of Plasmas, **6**, 751 (1999).

Cumming A. *et al.*, *Magnetic Field Evolution in Neutron Star Crusts Due to the Hall Effect and Ohmic Decay,* Astrophys. J., **609**, 999 (2004).

Decarreau A. *et al.*, *Formes canoniques des équations confluentes de l'équation de Heun*, Annales de la Societe Scientifique de Bruxelles, **92**, 53 (1978).

Duncan R.C., Thompson C., *Formation of Very Strongly Magnetized Neutron-Stars – Implications for Gamma – Ray Bursts*, Astrophys. J., **392**, L9 (1992).

Goldreich P., Reisenegger A., *Magnetic Field Decay in Isolated Neutron Stars*, Astrophys. J., **395**, 250 (1992).

Heun K., *Zur Theorie der Riemann'schen Functionen zweiter Ordnung mit Vier Verzweigungspunkten*, Math. Ann., **33**, 161 (1889).

Hollerbach R., Rudiger G., *Hall Drift in the Stratified Crusts of Neutron Stars,* Mon. Not. Roy. Astron. Soc., **347**, 1273 (2004).

Hoyos J. *et al.*, *Magnetic Field Evolution in Neutron Stars: One-Dimensional Multi-Fluid Model*, Astron. Astrophys., **487**, 789 (2008).

Kaplan D.B., Nelson A.F., *Strange Goings on in Dense Nucleonic Matter*, Phys. Lett. B, **175**, 57 (1986).
Mitsuda K. *et al*., *The X-Ray Observatory Suzaku*, PASJ, **59**, 1 (2007).
Pal S., Bandyopadhyay D., Greiner W., *Anti-K-Condensation in Neutron Stars*, Nucl. Phys. A, **674**, 553 (2000).
Slavyanov S.Y., Lay W., *Special Functions*, *A Unified Theory Based on Singularities*, Oxford Mathematical Monographs, Oxford (2000).
Thompson C., Beloborodov A.M., *High-Energy Emission from Magnetars*, Astrophys. J., **634**, 565 (2005).
Weber F., *From Boson Condensation to Quark Deconfinement: The Many Faces of Neutron Star Interiors*, Acta Phys. Polon. B, **30**, 3149 (1999).

SOLUȚII HEUN PENTRU BOZONI ÎNCĂRCAȚI ÎN CÂMP MAGNETIC CU SIMETRIE AXIALĂ

(Rezumat)

Această lucrare are în atenție o abordare analitică asupra evoluției bozonului relativist încărcat în crusta unui magnetar. Plecând de la ecuațiile Klein-Gordon, obținem funcția Heun Biconfluentă pentru un câmp magnetic static, intens, ortogonal pe un câmp electric radial.