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NOTES ON REVERSIBILITY AND BRANCHING OF GEODESICS IN FINSLER SPACES

BY

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Abstract. The existence conditions of reversible geodesics and *lpd*-symmetric curves are studied by using the notion of linear parallel displacement. Especially, local existence conditions of them are obtained by investigating their integrability conditions. Further, branching of geodesics is investigated.

Keywords: reversible geodesic; *lpd*-symmetric curve; linear parallel displacement; branching of geodesics; Finsler space.

Introduction

One of authors has been studying parallel displacements of vector fields along a curve from 2008 (Nagano, 2008; Nagano, 2010). The definition (Definition 0.1) is different from a traditional one (Aikou and Kozma, 2008). The most different point is the linearity of the differential equations with respect to a moved vector field. The linearity leads us to define a inner product of

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vector fields along a curve (Nagano, 2008; Nagano, 2010). Prof. Z. Shen, however, already had shown such definition satisfying the linearity in his book (Chern and Shen, 2005) by the coefficients N_j^i of an nonlinear connection N on TM. He called such vector fields "linearly parallel". Under a Finsler connection which Deflection tensor field $D_j^i := F_{rj}^i y^r - N_j^i$ and torsion tensor field $T_{rj}^i := F_{rj}^i - F_{jr}^i$ vanish, Definition 0.1 coincides with Z. Shen's one. Therefore we call our parallelism "linearly parallel" (Nagano, 2011)). The definition of linear parallel displacement is as follows (Nagano, 2008; Nagano, 2010; Nagano, 2011):

Definition 0.1 *For a curve $c(t) = (c^i(t))(a \leq t \leq b)$ on M and a vector field $v = (v^i(t))$ along c, if the equation*

$$\frac{dv^i}{dt} + F_{jr}^i(c, \dot{c})v^j \dot{c}^r = 0 \quad \left(\dot{c}^r = \frac{dc^r}{dt}\right) \tag{0.1}$$

is satisfied, then v is called a parallel vector field along c, and we call the linear map $\Pi_C : v(a) \to v(b)$ a linear parallel displacement along c.

Since Eq. (0.1) is linear with respect to v, the inverse vector field v^{-1} is not necessary parallel along the inverse curve c^{-1}, even if v is parallel vector field along a curve c. In studying about that the inverse vector field v^{-1} is also parallel, the conditions are obtained, and then, it is called "symmetric" or "lpd -symmetric" (Definition 1.1 and 1.3). In there, a following Finsler tensor field

$$H_j^i(x, y) := F_{rj}^i(x, y)y^r + F_{rj}^i(x, -y)(-y^r) = F_{0j}^i(x, y) + F_{0j}^i(x, -y) \tag{0.2}$$

plays important role.

Remark 0.1 *The definitional equation of the traditional notion of parallel displacement is as follows* (Aikou and Kozma, 2008):

$$\frac{dv^i}{dt} + N_r^i(c, v)\dot{c}^r = 0 \quad \left(\dot{c}^r = \frac{dc^r}{dt}\right) \tag{0.3}$$

Now, a geodesic in Finsler space exists unique at any point and any direction locally. This fact is the same as in Riemannian case. However the reverse curve of a geodesic is not necessary one. M. Crampin investigated about the reversibility of geodesics in Finsler space and obtained the conditions of it in (Crampin, 2005). Further in it he pointed two kind of geodesics out, which are "reversible" and "strictly reversible", for the first time in the history of Finsler geometry. We noticed that the reversible condition which M. Crampin had in (Crampin, 2005) is arranged to tensorial form by using H_j^i, and we investigate the integrability conditions for a geodesic by using the arranged conditions. In this paper, first we make the relation of symmetric vectors, lpd-symmetric curve and reversible geodesics clear (Section 1, Section 2). Second we study the existence conditions of reversible geodesic and lpd-symmetric curves (Section 3), and the last, behaviors of a geodesic passing through a one point along a direction are made clear, locally (Section 4).

We have some remarks here. In ordinary, the reverse curve c^{-1} and the reverse vector field v^{-1} of a curve $c(t)(a \leq t \leq b)$ and vector field v along $c(t)$ are defined by $c^{-1}(t) \coloneqq c(a+b-t)$ and $v^{-1}(t) \coloneqq v(a+b-t)$. However in this paper, for the revers parameter $\tau \coloneqq a+b-t$, these are defined by

$$c^{-1}(\tau) \coloneqq c(a+b-\tau) \quad and \quad v^{-1}(\tau) \coloneqq v(a+b-\tau) \tag{0.4}$$

Then $c(t)$, $c^{-1}(\tau)$ and $v(t)$, $v^{-1}(\tau)$ are always the same $c(t) = c^{-1}(\tau)$ and $v(t) = v^{-1}(\tau)$, however, the velocities $\dot{c} = \frac{dc}{dt}, \dot{c}^{-1} = \frac{dc^{-1}}{d\tau}$ are opposite, $\dot{c}^{-1} = -\dot{c}$.

This is very convenient for us in studying reversible geodesics and lpd-symmetric curves. In addition, according to (Matsumoto, 1986; Matsumoto, 2003) we put terminology and notations used in this paper as follows: Let M be an n-dimensional differentiable manifold and $x = (x^i)$ a local coordinate of M. TM is the tangent bundle of M and $(x,y) = (x^i, y^i)$ is a local coordinate of TM. $N = (N_j^i(x,y))$ is an nonlinear connection of TM and its coefficients of N on a local coordinate (x,y). $F(x,y)$ is a Finsler structure (or Finsler metric, Finsler fundamental function) on M. Further, $F\Gamma = (N_j^i(x,y), F_{jr}^i(x,y), C_{jr}^i(x,y))$ is a Finsler connection and its coefficients of $F\Gamma$ satisfying $T_{jr}^i = 0$, $D_j^i = 0$ and $g_{ij|k} = 0$ (h-metrical). Then $N_j^i(x,y), F_{jr}^i(x,y), C_{jr}^i(x,y)$ are positively homogeneous of degree $1, 0$ and -1, respectively, and N_j^i and F_{jr}^i come to Cartan's (Chern's or Rund's) ones. Further the nonlinear connection $G_j^i(x,y)$ of Berwald connection and $F_{jr}^i(x,y)$ satisfy $G_0^i(x,y)(= G^i(x,y)) = F_{00}^i(x,y)$. The condition $T_{rj}^i = 0$ is very important to the argument of Theorem 1.1 and $D_j^i = 0$ is needed to study of reversibility of geodesics under linear parallel displacement. Further the condition $g_{ij|k} = 0$ gives invariance of the inner product on a geodesic (Nagano, 2008; Nagano, 2010). Last, we denote the collection of horizontal vectors at every point on TM by $\mathcal{H}$. This is the subbundle of TTM and its dimension is $3n$. So we denote a local coordinate of $\mathcal{H}$ by (x,y,z). And it is called "horizontal subbundle of TTM". All of objects appeared in this paper (curves, vector fields, etc) are differentiable. In additions, indexes $a, b, c, \cdots, h, i, j, k, l, m, \cdots$ run on from 1 to $n = \dim M$.

1. Linear Parallel Displacement

Since F^i_{jr} in Eq. (0.1) has $y(= \dot{c})$, the reverse vector field v^{-1} is not necessary parallel vector field, in general. We so have the following notion

Definition 1.1 *For a parallel vector field v along a curve c, if the reverse vector field v^{-1} is also one along c^{-1}, then v is called a symmetric parallel vector field along c.*

Then we have

Theorem 1.1 *For a parallel vector field v along c, v is a symmetric parallel vector field along c if and only if the equation*

$$H^i_j(c, \dot{c})v^j = 0 \tag{1.1}$$

is satisfied on c.

Proof. Let $c(t) = (c^i(t))$ be a curve and $v(t) = (v^i(\mathrm{t}))$ a parallel vector field along $c(t)$. From the definition Eq. (0.4) of an inverse curve, $c^{-1}(\tau) = c(t), \dot{c}^{-1i}(\tau) = -\dot{c}^i(t), v^{-1}(\tau) = v(t)$ and $\dot{v}^{-1i}(\tau) = -\dot{v}^i(t)$ are satisfied. Then the inverse vector field $v^{-1}(\tau)$ of a parallel vector field $v(t)$ satisfies, from Eq. (0.1),

$$\frac{dv^{-1i}}{d\tau} + F^i_{jr}(c^{-1}, -\dot{c}^{-1})v^{-1j}\dot{c}^{-1r} = 0 \quad (\dot{c}^{-1r} = \frac{dc^{-1r}}{d\tau}). \tag{1.2}$$

We assume $H^i_j(c, \dot{c})v^j = 0$, then $F^i_{0j}(c, \dot{c})v^j = -F^i_{0j}(c, -\dot{c})v^j$ is satisfied, and from $T^i_{rj} = 0$, further $F^i_{0j}(c, \dot{c})v^j = F^i_{j0}(c, \dot{c})v^j$ is also true. So

$$\begin{aligned}\frac{dv^{-1i}}{d\tau} + F^i_{jr}(c^{-1}, -\dot{c}^{-1})v^{-1j}\dot{c}^{-1r} &= \frac{dv^{-1i}}{d\tau} + F^i_{rj}(c^{-1}, -\dot{c}^{-1})v^{-1j}\dot{c}^{-1r}\\ &= \frac{dv^{-1i}}{d\tau} - F^i_{0j}(c^{-1}, -\dot{c}^{-1})v^{-1j}\\ &= \frac{dv^{-1i}}{d\tau} + F^i_{0j}(c^{-1}, \dot{c}^{-1})v^{-1j}\\ &= \frac{dv^{-1i}}{d\tau} + F^i_{rj}(c^{-1}, \dot{c}^{-1})v^{-1j}\dot{c}^{-1r}\\ &= \frac{dv^{-1i}}{d\tau} + F^i_{jr}(c^{-1}, \dot{c}^{-1})v^{-1j}\dot{c}^{-1r}\end{aligned} \tag{1.3}$$

is satisfied. And when the left hand side vanishes, the right hand side also vanishes. Therefore v^{-1} is parallel vector field along c^{-1}. Inversely, if $\frac{dv^i}{dt} + N^i_r(c,v)\dot{c}^r = 0$ and $\frac{dv^{-1i}}{d\tau} + F^i_{jr}(c^{-1}, -\dot{c}^{-1})v^{-1j}\dot{c}^{-1r} = 0$ are satisfied, then $F^i_{0j}(c,\dot{c})v^j = -F^i_{0j}(c,-\dot{c})v^j$ is also true. *Q.E.D.*

Furthermore we put a following definition

Definiton 1.2 *For a parallel vector field v along a curve c, if the following equation*

$$H^i_j(c,\dot{c})v^j = \phi(c,\dot{c})v^i \tag{1.4}$$

is satisfied, then v is called a quasi symmetric parallel vector field along c, where ϕ is a scalar function on $(c,\dot{c})$ with $\phi(c,-\dot{c}) = \phi(c,\dot{c})$, namely, absolutely homogeneous of degree 1 *in $\dot{c}$.*

Remark 1.1 *When v is a quasi symmetric parallel vector field along c, the reverse vector field v^{-1} satisfies*

$$\frac{dv^{-1i}}{d\tau} + F^i_{jr}(c^{-1}, -\dot{c}^{-1})v^{-1j}\dot{c}^{-1r} = \phi(c^{-1}, -\dot{c}^{-1})v^{-1i} \tag{1.5}$$

Last, we put following definitions

Definition 1.3 (1) *If any parallel vector field v is necessary symmetric parallel vector field along a curve c, then the linear parallel displacement Π_C is called symmetric and the curve c is called lpd-symmetric.*
(2) *If any parallel vector field v is necessary quasi symmetric parallel vector field along a curve c, then the linear parallel displacement Π_C is called quasi symmetric and the curve c is called lpd-quasi symmetric.*

Then we have

Proposition 1.1 *A curve c is lpd-symmetric (or lpd-quasi symmetric) if and only if the following equation*

$$H^i_j(c,\dot{c}) = 0 \ \ (or\ \ H^i_j(c,\dot{c}) = \phi(c,\dot{c})\delta^i_j) \tag{1.6}$$

is satisfied on c, where ϕ is a scalar function which is absolutely homogeneous of degree 1 *in $\dot{c}$.*

(∵) From the arbitrariness of v in Eq. (1.1) and Eq. (1.4), above theorem is right. ■

Remark 1.2 *If* $F_{rj}^{i}(x,y) = F_{rj}^{i}(x,-y)$ *is satisfied, then, on any curve* c, $H_j^i(c,\dot{c}) = 0$ *is true. So, in Riemannian and Berwald spaces, any linear parallel displacement* Π_C *is symmetric and any curvec is* lpd*-symmetric.*

Further, since H_j^i is a Finsler tensor field, we have

Proposition 1.2 *Let* $F\Gamma$ *be a Finsler connection with Cartan's* $N_j^i(x,y)$ *and* $F_{jr}^{i}(x,y)$*. Then any curve in a Finsler space* $(M, F, F\Gamma)$ *is* lpd*-symmetric (or* lpd*-quasi symmetric) if and only if* $H_j^i(x,y)$ *vanishes, namely,*

$$H_j^i(x,y) = 0 \tag{1.7}$$

is satisfied (or $H_j^i(x,y) = \phi(x,y)\delta_j^i$ *is satisfied with a scalar function* $\phi(x,y)$ *on* TM *which is absolutely homogeneous of degree* 1 *in* y*).*

Remark 1.3 (1) *From Definition* 1.3 (1) *and the equation Eq.* (1.1)*, we can see that a symmetric parallel vector field* v *along* c *is the eigenvector of an eigenvalue* 0 *at every point on* c.
(2) *From Definition* 1.3 (2) *and the equation Eq.* (1.4)*, we can see that a quasi symmetric parallel vector field* v *along* c *is the eigenvector of a real eigenvalue* ϕ *at every point on* c.

2. Reversible Geodesics

We state the notion of reversible geodesics introduced by M. Crampin (Crampin, 2005) and some properties of them.

Definition 2.1 *For a curve* $c(s)(s\colon \mathit{the\ arc-length})$*, if* c *is locally a distance-minimizing curve and a critical point of the functional* $\mathcal{F}_F$ *as follows:*

$$\mathcal{F}_F\colon c(s) \in \Gamma(p,q) \to \mathcal{F}_F(c) = \int_c F(c,\dot{c})\, ds \in \mathcal{R}^+ \ \left(\dot{c} = \frac{dc}{ds}\right), \tag{2.1}$$

where $\Gamma(p,q)$ *is a set of smooth oriented regular curves with the endpoints* p *and* q*, then* c *is called a geodesic from* p *to* q.

Remark 2.1 *We know very well that a geodesic* $c(s) = (c^i(s))$ *satisfies the following equation*

$$\frac{d^2c^i}{dt^2} + F_{jr}^{i}(c,\dot{c})\dot{c}^j\dot{c}^r = 0. \tag{2.2}$$

In (Crampin, 2005)*, the geodesic equation is written by the coefficient* $G_j^i(x,y)$ *of Berwald connection as follows:*

$$\frac{d^2c^i}{dt^2} + G^i(c,\dot{c}) = 0. \quad (2.3)$$

However, these are equivalent because of $G^i(x,y) = G^i_{jr}(x,y)y^j y^r = F^i_{jr}(x,y)y^j y^r$.

Definition 2.2 (Crampin, 2005) *If a curve* c *is a geodesic and the reverse curve* c^{-1} *is also one, then* c *is called a reversible geodesic.*

Remark 2.2 *The reverse parameter* $\tau(= l - s)$ *is not necessary affine one of* c^{-1}*, where* l *is the length of* c.

Theorem 2.1 *A geodesic* $c(s) = (c^i(s))$ *is reversible if and only if there is a certain scalar function* $\phi(c,\dot{c})$ *satisfying the following equation*

$$H^i_0(c,\dot{c}) \coloneqq H^i_j(c,\dot{c})\dot{c}^j = \phi(c,\dot{c})\dot{c}^i, \quad (2.4)$$

where $\phi(c,\dot{c})$ *is absolutely homogeneous of degree* 1 *in* $\dot{c}$.

Proof. We assume the equation Eq. (2.4). Then the reverse curve $c^{-1}(\tau) = (c^{-1i}(\tau))$ satisfies

$$\frac{d^2c^{-1i}}{d\tau^2} + F^i_{jr}(c^{-1},\dot{c}^{-1})\dot{c}^{-1j}\dot{c}^{-1r} = \phi(c^{-1},\dot{c}^{-1})\dot{c}^{-1i} \quad (2.5)$$

where $\tau = l - s$, $\dot{c}^{-1i} = -\dot{c}^i$, $\frac{d^2c^{-1i}}{d\tau^2} = \frac{d^2c^i}{ds^2}$.
We consider a transformation $\sigma \to \tau = \sigma(\tau)$ of the parameter. Then

$$\frac{d^2c^{-1i}}{d\sigma^2} + F^i_{jr}\left(c^{-1}(\sigma),\dot{c}^{-1}(\sigma)\right)\dot{c}^{-1j}(\sigma)\dot{c}^{-1r}(\sigma) = \\ = \left(\frac{d^2\tau}{d\sigma^2} + \phi\left(c^{-1}(\tau),\dot{c}^{-1}(\tau)\right)(\frac{d\tau}{d\sigma})^2\right)\frac{d\dot{c}^{-1i}}{d\tau}$$

is satisfied. We can get a solution of the ordinary differential equation $\frac{d^2\tau}{d\sigma^2} + \phi(\tau)(\frac{d\tau}{d\sigma})^2 = 0$, easily. It is

$$\sigma = a\int_0^\tau e^{\int_0^\gamma \phi(\rho)d\rho}\,d\gamma, (a > 0\colon constant).$$

This solution leads c^{-1} is geodesic. Inversely, if both of $c(s)$ and $c^{-1}(\tau)$ are geodesics, then Eq. (2.4) is satisfied, obviously. *Q.E.D.*

Remark 2.3 (1) *The necessary and sufficient condition of reversibility are introduced firstly by M. Crampin* (Crampin, 2005) *as the following equation*

$$G^i(c,-\dot{c}) = G^i(c,\dot{c}) + \lambda(c,\dot{c})\dot{c}^i, \tag{2.6}$$

where $\lambda = \lambda(x,y)$ *is a function, which is an absolutely homogeneous of degree* 1 *in* y.
(2) *Our case is* $\phi(x,y) = -\lambda(x,y)$.

Next, we state the notion of "strictly reversible".

Definiton 2.3 (Crampin, 2005) *For a reversible geodesic* $c(s)$, *if the parameter* τ *of the reverse geodesic* $c^{-1}(\tau)$ *is an affine parameter, namely,*

$$\tau = a\bar{s} \;\; (a > 0{:}\, constant), \tag{2.7}$$

then c *is called strictly reversible.*

Remark 2.4 (1) *The original definition of strictly reversible in* (Crampin, 2005) *is* $\lambda = 0$. *That is*

$$G^i(c,\dot{c}) = G^i(c,-\dot{c}). \tag{2.8}$$

(2) *The reverse geodesic* $c^{-1}(\tau) = (c^{-1i}(\tau))$ *satisfies*

$$\frac{d^2c^{-1i}}{d\tau^2} + F^i_{jr}(c^{-1},\dot{c}^{-1})\dot{c}^{-1j}\,\dot{c}^{-1r} = 0 \qquad .(2.9)$$

Theorem 2.2 A geodesic c is strictly reversible if and only if the equation

$$H^i_0(c,\dot{c}) = 0 \tag{2.10}$$

is satisfied.
($\because$) From Eq. (2.7) and $\phi = -\lambda$, it is trivial. ■

From Proposition 1.1, Theorem 2.1 and 2.2, we have

Proposition 2.1 (1) *If a geodesic* c *is* lpd*-symmetric, then* c *is strictly reversible.*
(2) *If a geodesic* c *is* lpd*-quasi symmetric, then* c *is reversible*.

Further from Definition 1.2, Theorem 1.1, 2.1 and 2.2, we have

Remark 2.5 *For a geodesic c and its affine parameter,*
(1) *c is strictly reversible if and only if $\dot{c}$ is a symmetric parallel vector field along c.*
(2) *c is reversible if and only if $\dot{c}$ is quasi symmetric parallel vector field along c.*

3. Existence Conditions

In Section 1 and 2, we have the necessary and sufficient conditions for a curve c to be lpd-symmetric, reversible or strictly reversible geodesic. In this section, we study the necessary and sufficient conditions of existing of these curves around at any point and direction.

First, we treat the existence condition of lpd-symmetric curves.

For a curve c, it is lpd-symmetric if and only if $H_j^i(c, \dot{c}) = 0$ is satisfied (Proposition 1.2). Therefore for any point x and a direction y at x, according to the general theory of differential equations, there exists a lpd-symmetric curve $c(t)$ satisfying $c(0) = x$ and $\dot{c}(0) = y$ if and only if

$$\frac{dH_j^i}{dt}(c, \dot{c}) = \frac{\partial H_j^i}{\partial x^k}(c, \dot{c})\dot{c}^k + \frac{\partial H_j^i}{\partial y^k}(c, \dot{c})\ddot{c}^k = 0 \tag{3.1}$$

is satisfied on a certain neighborhood of (x, y). From the arbitrariness of $\ddot{c}^k$, we have $\frac{\partial H_j^i}{\partial x^k}(c, \dot{c})\dot{c}^k = 0$ and $\frac{\partial H_j^i}{\partial y^k}(c, \dot{c}) = 0$. And from the homogeneous property of degree 1 in y $\left(\frac{\partial H_j^i}{\partial y^k} y^k = H_j^i\right)$ of the Finsler tensor field $H_j^i(x, y)$, we notice $\frac{\partial H_j^i}{\partial y^k}(x, y) = 0$ is equivalent to $H_j^i(x, y) = 0$.

In addition, by using the similar observation we have $H_j^i - \phi\delta_j^i = 0$ for lpd-quasi symmetric curve. Then we have

Proposition 3.1 *For any point x and a direction y at x, there exists a lpd-symmetric curve $c(t)$ satisfying $c(0) = x$ and $\dot{c}(0) = y$ if and only if*

$$H_j^i(x, y) = 0 \tag{3.2}$$

is satisfied, and there exists a lpd-quasi symmetric curve $c(t)$ satisfying $c(0) = x$ and $\dot{c}(0) = y$ if and only if

$$H_j^i(x, y) = \phi(x, y)\delta_j^i \tag{3.3}$$

is satisfied, where $\phi(x, y)$ is a local function on a certain neighborhood of (x, y).

Next, we consider the existence condition of strictly reversible geodesics.

From Theorem 2.2, a geodesic $c(t)$ is strictly reversible if and only if $H_0^i(c,\dot{c}) = 0$ is satisfied. So we study the integrability condition of the following differential equation system

$$\begin{cases} \ddot{c}^i + F_{00}^i(c,\dot{c}) = 0 \\ H_0^i(c,\dot{c}) = 0 \end{cases} \tag{3.4}$$

Then, according to the theory of differential equation, the integrability condition for $c(t)$ at a point $c(0) = x$ and a direction $\dot{c}(0) = y$ is that the following equation

$$\frac{dH_0^i}{dt}(c,\dot{c}) = \frac{d}{dt}\left(H_j^i(c,\dot{c})\dot{c}^j\right) = \frac{\partial H_j^i}{\partial x^k}(c,\dot{c})\dot{c}^k\dot{c}^j + \frac{\partial H_j^i}{\partial y^k}(c,\dot{c})\ddot{c}^k\dot{c}^j + \\ + H_j^i(c,\dot{c})\ddot{c}^j = 0 \tag{3.5}$$

is satisfied on a certain neighborhood of (x,y).

From $\ddot{c}^i = -F_{00}^i(c,\dot{c})$ and $N_0^k(c,\dot{c}) = F_{00}^k(c,\dot{c})$, Eq. (3.5) is rewritten as follows:

$$\frac{dH_0^i}{dt}(c,\dot{c}) = \frac{\delta H_j^i}{\delta x^k}(c,\dot{c})\dot{c}^k\dot{c}^j - H_k^i(c,\dot{c})F_{00}^k(c,\dot{c}) \tag{3.6}$$

In general, under $H_0^i(x,y) = 0$ and $N_0^k(x,y) = F_{00}^k(x,y)$, the quantity

$$K_{jrk}^i(x,y,z) \coloneqq \frac{\delta F_{jr}^i}{\delta x^k}(x,y) - \frac{\delta F_{jk}^i}{\delta x^r}(x,z) - F_{mr}^i(x,y)F_{jk}^m(x,y) + \\ + F_{mk}^i(x,z)F_{jr}^m(x,z)$$ (Nagano, 2012; Nagano, 2013) satisfied

$$K_{jrk}^i(x,y,-y)y^j y^r y^k = \frac{\delta H_j^i}{\delta x^k}(x,y)y^k y^j. \tag{3.7}$$

$(\because)$

$$\frac{\delta H_j^i}{\delta x^k}(x,y) =$$
$$= \frac{\delta F_{jr}^i}{\delta x^k}(x,y)y^r - \frac{\delta F_{jk}^i}{\delta x^k}(x,-y)y^r - F_{mj}^i(x,y)F_{rk}^m(x,y)y^r + \\ + F_{mj}^i(x,-y)F_{rk}^m(x,-y)y^r.$$

On the other hand,

$$K_{jrk}^{i}(x,y,-y) \coloneqq \frac{\delta F_{jr}^{i}}{\delta x^{k}}(x,y) - \frac{\delta F_{jk}^{i}}{\delta x^{r}}(x,-y) - F_{mr}^{i}(x,y)F_{jk}^{m}(x,y) + F_{mk}^{i}(x,-y)F_{jr}^{m}(x,-y)$$

$\frac{\delta H_j^i}{\delta x^k}(x,y)$ is different from $K_{jrk}^{i}(x,y,-y)$ but by contractions of y^j, y^k and y^r, Eq. (3.7) is satisfied. ■

So from Eq. (3.6) and Eq. (3.7), we have

$$\frac{dH_0^i}{dt}(c,\dot{c}) = K_{jrk}^{i}(c,\dot{c},-\dot{c})\dot{c}^j\dot{c}^r\dot{c}^k - H_k^i(c,\dot{c})F_{00}^k(c,\dot{c}). \tag{3.8}$$

From the integrability condition Eq. (3.5) and Eq. (3.8)

$$K_{jrk}^{i}(x,y,-y)y^j y^r y^k = H_k^i(x,y)F_{00}^k(x,y) \tag{3.9}$$

is satisfied.

On the other hand, we have an normal coordinate around any point (Busemann, 1955). It is that the equation of a geodesic is written by a rectilinear equation. That means $F_{00}^i(x,y) = 0$ because of $\ddot{c}^i + F_{00}^i(c,\dot{c}) = 0$. Of course, the origin of it is not of class C^2, but that is not obstruction because we consider a domain whose (x,y) is a point close to the origin of the normal coordinate. Therefore we can assume $F_{00}^k(x,y) = 0$, and the tensorial property of $K_{jrk}^{i}(x,y,z)$change Eq. (3.9) to

$$K_{jrk}^{i}(x,y,-y)y^j y^r y^k = 0. \tag{3.10}$$

This is the integrability condition for Eq.(3.4).

Next is the existence condition of reversible geodesics.

From Theorem 2.1, a geodesic c is reversible if and only if there is a scalar function satisfying $\phi(\mathrm{c},\dot{c}) = \phi(\mathrm{c},-\dot{c})$and $H_j^i(\mathrm{c},\dot{c})\dot{c}^j = \phi(\mathrm{c},\dot{c})\dot{c}^i$. So we consider the integrability condition of the following differential equation system

$$\begin{cases} \ddot{c}^i + F_{00}^i(c,\dot{c}) = 0 \\ H_0^i(c,\dot{c}) = \phi(c,\dot{c})\dot{c}^i \end{cases} \tag{3.11}$$

Then, according to the theory of differential equation, the condition of integrability for $c(t)$ at any point $c(0) = x$ and a direction $\dot{c}(0) = y$ is that the following equation

$$\frac{d}{dt}\left(H_0^i(c,\dot{c}) - \phi(\mathrm{c},\dot{c})\dot{c}^i\right) = 0 \tag{3.12}$$

is satisfied. We have

$$\frac{d}{dt}\left(H_0^i(c,\dot{c}) - \phi(\mathrm{c},\dot{c})\dot{c}^i\right) = \\ = \frac{\partial H_j^i}{\partial x^k}(c,\dot{c})\dot{c}^k\dot{c}^j + \frac{\partial H_j^i}{\partial y^k}(c,\dot{c})\ddot{c}^k\dot{c}^j + H_j^i(c,\dot{c})\ddot{c}^j - \\ -\frac{\partial \phi}{\partial x^k}(c,\dot{c})\dot{c}^i\dot{c}^k - \frac{\partial \phi}{\partial y^k}(c,\dot{c})\ddot{c}^k\dot{c}^i - \phi(\mathrm{c},\dot{c})\ddot{c}^i. \tag{3.13}$$

And we can arrange Eq. (3.13) by $\ddot{c}^i = -F_{00}^i(c,\dot{c}), \frac{\partial H_j^i}{\partial x^k} = \frac{\delta H_j^i}{\delta x^k} + N_k^m \frac{\partial H_j^i}{\partial y^m}, \frac{\partial \phi}{\partial x^k} = \frac{\delta \phi}{\delta x^k} + N_k^m \frac{\partial \phi}{\partial y^m}$ *and* $N_0^i(c,-\dot{c}) = N_0^i(c,\dot{c}) - \phi(\mathrm{c},\dot{c})\dot{c}^i,\ F_{00}^i(c,\pm\dot{c}) = N_0^i(c,\pm\dot{c})$ as follows:

$$\frac{d}{dt}\left(H_0^i(c,\dot{c}) - \phi(\mathrm{c},\dot{c})\dot{c}^i\right) = \\ = \frac{\delta H_j^i}{\delta x^k}(c,\dot{c})\dot{c}^k\dot{c}^j - \frac{\delta \phi}{\delta x^k}(c,\dot{c})\dot{c}^i\dot{c}^k - \left(H_k^i(c,\dot{c}) - \phi(c,\dot{c})\delta_k^i\right)F_{00}^k(c,\dot{c}). \tag{3.14}$$

From Eq. (3.14), the integrability condition Eq. (3.12) is rewritten to

$$K_{jrk}^i(x,y,-y) = \frac{\delta \phi}{\delta x^k}(x,y)y^i y^k + \left(H_k^i(x,y) - \phi(x,y)\delta_k^i\right)F_{00}^k(x,y) \tag{3.15}$$

From the existence of an normal coordinate around (x,y) and the tensorial property of $K_{jrk}^i(x,y,z)$, we have

$$K_{jrk}^i(x,y,-y)y^j y^r y^k = \frac{\delta \phi}{\delta x^k}(x,y)y^i y^k. \tag{3.16}$$

This is the integrability condition for Eq. (3.11).

Last is the existence condition of lpd-symmetric and geodesic curves.

We see if a curve is lpd-symmetric and geodesic then it is strictly reversible from Proposition 2.1. The condition that a curve $c(t)$ is lpd-symmetric is $H_j^i(c,\dot{c}) = 0$. So the PDE which we should study is as follows:

$$\begin{cases} \ddot{c}^i + F_{00}^i(c,\dot{c}) = 0 \\ H_j^i(c,\dot{c}) = 0. \end{cases} \tag{3.17}$$

The integrability condition of Eq. (3.17) is $\frac{dH_j^i}{dt}(c,\dot{c}) = 0$ but because $c(t)$ is a geodesic, from $\frac{dH_j^i}{dt}(c,\dot{c}) = \frac{\partial H_j^i}{\partial x^k}(c,\dot{c})\dot{c}^k + \frac{\partial H_j^i}{\partial y^k}(c,\dot{c})\ddot{c}^k$ and $\ddot{c}^i = -F_{00}^i(c,\dot{c})$, it is as follows:

$$\frac{\delta H_j^i}{\delta x^k}(x,y)y^k = 0 \tag{3.18}$$

is satisfied on a certain neighborhood of (x,y).

In general, $\frac{\delta H_j^i}{\delta x^k}(x,y)y^k$ and $K_{jrk}^i(x,y,-y)y^r y^k$ have a relation as follows:

$$\begin{aligned} &K_{jrk}^i(x,y,-y)y^r y^k + H_m^i(x,y)F_{0j}^m(x,y) = \\ &= \frac{\delta H_j^i}{\delta x^k}(x,y)y^k + (F_{mj}^i(x,y) - F_{mj}^i(x,-y))F_{00}^m(x,y) \end{aligned} \tag{3.19}$$

Then, this is changed as follows:

$$\begin{aligned} K_{jrk}^i(x,y,-y)y^r y^k = \left(F_{mj}^i(x,y) - F_{mj}^i(x,-y)\right)F_{00}^m(x,y) - \\ -H_m^i(x,y)F_{0j}^m(x,y). \end{aligned} \tag{3.20}$$

There exists a geodesic coordinate around any point (x,y), which the coefficient $F_{rj}^i(x,y)$ vanishes on it. Therefore we have, from Eq. (3.20),

$$K_{jrk}^i(x,y,-y)y^r y^k = 0. \tag{3.21}$$

Finally, we have a following theorem:

Theorem 3.1 *For any point x and a direction y at x,*
(1) *there exists a strictly reversible geodesic $c(t)$ satisfying $c(0) = x, \dot{c}(0) = y$ if and only if*

$$K_{jrk}^i(x,y,-y)y^j y^r y^k = 0 \tag{3.22}$$

is satisfied on a certain neighborhood of (x,y), and

(2) *there exists a reversible geodesic $c(t)$ satisfying $c(0) = x, \dot{c}(0) = y$ if and only if there is a scalar function $\phi(x,y)$ with $\phi(x,y) = \phi(x,-y)$ and the following equation*

$$K_{jrk}^i(x,y,-y)y^j y^r y^k = \frac{\delta\phi}{\delta x^k}(x,y)y^i y^k \tag{3.23}$$

is satisfied on a certain neighborhood of (x,y), and

(3) *there exists a lpd-symmetric and geodesic $c(t)$ satisfying $c(0) = x, \dot{c}(0) = y$ if and only if*

$$K^i_{jrk}(x, y, -y)y^j y^r = 0 \quad (3.24)$$

is satisfied on a certain neighborhood of (x, y).

Remark 3.1 *Theorem* 3.1(3) *is very interesting. If the space* $(M, F, F\Gamma)$ *is Riemannian, then* $K^i_{jrk}(x, y, z)$ *comes to Riemannian curvature* $R^i_{jrk}(x)$. *In a Riemannian space, Eq.* (3.24) *is trivial and there always exists a* lpd*-symmetric and geodesic curves passing through an arbitrary point* x. *Namely, a geometrical meaning of* $R^i_{jrk}(x)y^j y^r = 0$ *is noticed.*

4. Branching of Geodesics

We investigated the behavior of geodesics in a Finsler surface in (Innami *et al.*, 2016). In this section we discuss about branching of geodesics passing through one point. According to obtained results by studying of lpd-symmetric curves and reversible geodesics, the velocity of a reversible geodesic is symmetric or quasi symmetric vector at every point on the geodesic (Remark 2.5). A geodesic always exists at a point x and a direction $y(\in T_x M)$ locally. However, at x, a geodesic going along a direction y is different from one going along the reverse direction $-y$, in general. See Fig. 1(1). Let c be a geodesic going along a direction y and $\bar{c}$ another one going along $-y$ at x. Even if c is extended backward, however, the image does not always coincide with $\bar{c}$. See Fig. 1(2), c and $\bar{c}$ are the same as above (1). In this time, the image of backward extension of c coincides with $\bar{c}$ (c is reversible). In the last case Fig. 1(3), a reversible geodesic till x divides into c and $\bar{c}$ after x.

If the eigenpolynomial of the matrix $H^i_j(c(t_0), \dot{c}(t_0))$ has eigenvalue 0 at $c(t_0)$ and $\dot{c}(t_0)$ is the eigenvector of 0, then $H^i_0\big(c(t_0), \dot{c}(t_0)\big) = 0$ is satisfied. And if the eigenpolynomial of the matrix $H^i_j(c(t_0), \dot{c}(t_0))$ has real eigenvalue ϕ at $c(t_0)$ and $\dot{c}(t_0)$ is the eigenvector of ϕ, then $H^i_0\big(c(t_0), \dot{c}(t_0)\big) = \phi(c(t_0), \dot{c}(t_0))\dot{c}^i(t_0)$ is satisfied. Then we have

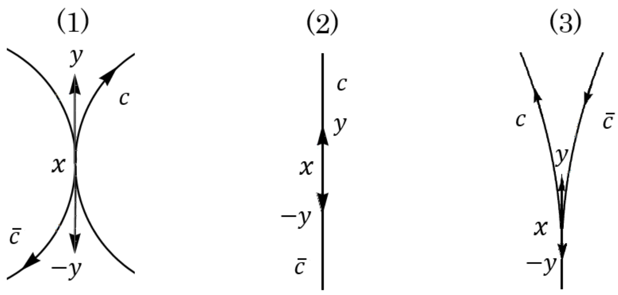

Fig. 1 – Branching of geodesic.

Theorem 4.1 *Let $c(t)$ be a geodesic going along a direction y and $\bar{c}(t)$ another one going along $-y$ at x.*
For a point $x = c(t_0) = \bar{c}(t_0)$,
(1) *if $H_0^i(c(t_0), \dot{c}(t_0)) \neq 0$ and $H_0^i(c(t_0), \dot{c}(t_0)) \neq \phi(c(t_0), \dot{c}(t_0))\dot{c}^i(t_0)$ are satisfied around x on c, then geodesics c and $\bar{c}$ divide into two branches in front and behind at x* (cf. Fig.1(1)), *or*
(2) *if $H_0^i(c(t_0), \dot{c}(t_0)) = 0$ or $H_0^i(c(t_0), \dot{c}(t_0)) = \phi(c(t_0), \dot{c}(t_0))\dot{c}^i(t_0)$ are satisfied around x on c, then c is reversible geodesic passing through x* (cf. Fig.1(2)), *or*
(3) *if $H_0^i(c(t_0), \dot{c}(t_0)) = 0$ or $H_0^i(c(t_0), \dot{c}(t_0)) = \phi(c(t_0), \dot{c}(t_0))\dot{c}^i(t_0)$ are satisfied till x on c and after x, c does not satisfy above conditions, then c and $\bar{c}$ divide into two branches after x* (see Fig. 1(3)).

5. Examples

We show three concrete examples for three cases of Theorem 4.1. The following situations are common on three examples.

Situation
$M(\subset \mathcal{R}^2)$ is an open domain, (x, y) is the coordinate of M, $(\dot{x}, \dot{y})$ is the coordinate of $T_{(x,y)}M$ and $(x, y, \dot{x}, \dot{y})$ is the coordinate of TM

(A) The case of Theorem 4.1(1) (See Fig. 2)
Let $\alpha^2 = dx \otimes dx + dy \otimes dy$ be a 2-form and $\beta = -y dx$ $(d\beta \neq 0)$ 1-form, and $F = \alpha + \beta$ is Randers metric (a special Finsler metric). Then

$$F(x, y, \dot{x}, \dot{y}) = \sqrt{\dot{x}^2 + \dot{y}^2} - y\dot{x} \tag{5.1}$$

is satisfied on TM. M is the band $\{(x, y) | -1 < y < 1\}$, and immediately, by using Euler-Lagrange equation, we have the equation of geodesic as follows

$$\ddot{x} = \dot{y}, \;\; \ddot{y} = -\dot{x}, \tag{5.2}$$

where $\dot{x}^2 + \dot{y}^2 = 1$ is assumed. Then, geodesic $c(t)$ is a circular arc as follows

$$c(t) = \begin{cases} x(t) = a\,cost + b\,sint + c_1 \\ y(t) = b\,cost - a\,sin\,t + c_2\,, \end{cases} \quad ((x - c_1)^2 + (y - c_2)^2 = 1), \tag{5.3}$$

where a, b, c_1, c_2 are constants and $a^2 + b^2 = 1$ is satisfied. The parameter t is not the arc-length or its affine parameter. The eigenvalues ϕ_1, ϕ_2 of $H_j^i(x, y, \dot{x}, \dot{y})$ are

$$\phi_1, \phi_2 = \frac{3\dot{x}\dot{y}(y^2+1) \pm \sqrt{16\dot{x}^4(y^2-1) + \dot{x}^2(9y^4+26y^2-15)\dot{y}^2 + 8y^2\dot{y}^4}}{2(y^2\dot{x}^2-1)}. \tag{5.4}$$

These ϕ_1, ϕ_2 don't satisfy $H_0^1(x, y, \dot{x}, \dot{y}) = \phi_r \dot{x}$, $H_0^2(x, y, \dot{x}, \dot{y}) = \phi_r \dot{y}$ $(r = 1,2)$ on $c(t)$.

Therefore $c(t)$ is not reversible. See Fig. 2, at origin O, a geodesic $c(t)$ has a velocity $v = (0,1)$ and $\bar{c}$ has a velocity $-v = (0, -1)$. Of course, at any point, the same state appeared.

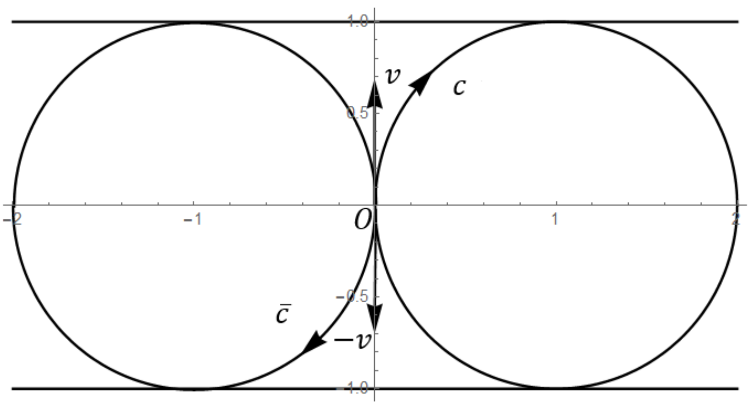

Fig. 2 – Branching of geodesic (not reversible).

(B) The case of Theorem 4.1(2) (See Fig. 3)

Let $\alpha^2 = dx \otimes dx + dy \otimes dy$ be a 2-form and $\beta = -ydy$ $(d\beta = 0)$) 1-form, and $F = \alpha + \beta$ is Randers metric. Then

$$F(x, y, \dot{x}, \dot{y}) = \sqrt{\dot{x}^2 + \dot{y}^2} - y\dot{y} \tag{5.5}$$

is satisfied on TM. M is the band $\{(x, y) | -1 < y < 1\}$, and immediately, by using Euler-Lagrange equation, we have the equation of geodesic as follows

$$\ddot{x} = 0, \ \ddot{y} = 0, \tag{5.6}$$

where $\dot{x}^2 + \dot{y}^2 = 1$ is assumed. Then, geodesic is a straight line as follows

$$c(t) = \begin{cases} x(t) = at + b \\ y(t) = ct + d, \end{cases} (c(x - b) - a(y - d) = 0), \tag{5.7}$$

where a, b, c, d are constants and $a^2 + c^2 = 1$ is assumed. The eigenvalues ϕ_1, ϕ_2 of $H_j^i(x, y, \dot{x}, \dot{y})$ are

$$\phi_1 = \frac{-2\dot{y}^2}{1-y^2\dot{y}^2},\ \phi_2 = \frac{-\dot{y}^2}{1-y^2\dot{y}^2}, \tag{5.8}$$

and only ϕ_1 always satisfy $H_0^1(x, y, \dot{x}, \dot{y}) = \phi_1 \dot{x},\ H_0^2(x, y, \dot{x}, \dot{y}) = \phi_1 \dot{y}$ on $c(t)$. Therefore all geodesics are reversible. In particular, straight lines $x = \pm t + b, y = d(\in \mathcal{R})$ are strictly reversible because $\dot{y} = 0$ leads to $\phi_1 = 0$ (Fig. 3).

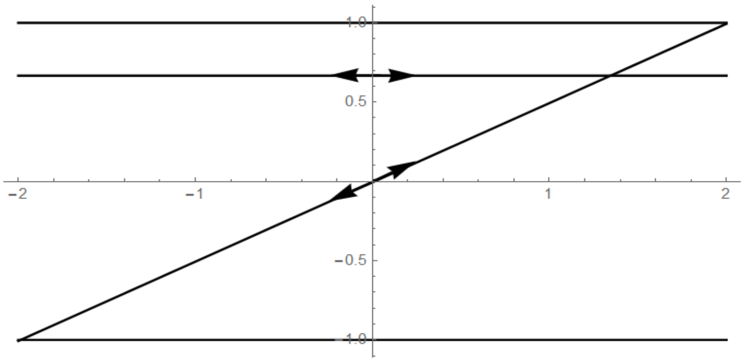

Fig. 3 – Branching of geodesic (reversible).

(C) The case of Theorem 4.1(3) (See Fig. 4)

Let $\alpha^2 = dx \otimes dx + dy \otimes dy$ be a 2-form and $\beta = -e(y)dx$ 1-form, and $F = \alpha + \beta$ is Randers metric, where $e(y) = \begin{cases} e^{-\frac{1}{y}} & (y > 0) \\ 0 & (y \leq 0). \end{cases}$

This $e(y)$ is C^∞-class, however, not ω-class at $y = 0$.

Then

$$F(x, y, \dot{x}, \dot{y}) = \sqrt{\dot{x}^2 + \dot{y}^2} - e(y)\dot{x} \tag{5.9}$$

is satisfied on TM. M is $\mathcal{R}^2$. In the area of $y \leq 0$, F is just Euclidean metric, so geodesics are straight lines and strictly reversible, of course. In the area of $y > 0$,

$$F(x, y, \dot{x}, \dot{y}) = \sqrt{\dot{x}^2 + \dot{y}^2} - e^{-\frac{1}{y}}\dot{x} \tag{5.10}$$

From Euler-Lagrange equation of F, we have

$$\ddot{x} = \frac{e^{-\frac{1}{y}}}{y^2}\dot{y}, \quad \ddot{y} = -\frac{e^{-\frac{1}{y}}}{y^2}\dot{x}, \tag{5.11}$$

where $\dot{x}^2 + \dot{y}^2 = 1$ is assumed. Then

$$\dot{x} = e^{-\frac{1}{y}} + a\ (a\!: constant), \quad \dot{y} = \pm\sqrt{1 - (e^{-\frac{1}{y}} + a)^2} \tag{5.12}$$

is obtained.

C-(I) A geodesic $x = f_1(y)$ having the velocity $(\dot{x}, \dot{y}) = (0,1)$ at Origin.

The equation is given as follows

$$\dot{x} = e^{-\frac{1}{y}},\ \dot{y} = \sqrt{1 - e^{-\frac{2}{y}}} \tag{5.13}$$

Then, from

$$\frac{dx}{dy} = \frac{\dot{x}}{\dot{y}} = \frac{e^{-\frac{1}{y}}}{\sqrt{1-e^{-\frac{2}{y}}}} \tag{5.14}$$

by using Taylor expansion, we have its image in Fig. 4.

C-(II) A geodesic $x = f_2(y)$ having the velocity $(\dot{x}, \dot{y}) = (0, -1)$ at Origin.

The equation is given as follows

$$\dot{x} = e^{-\frac{1}{y}},\ \dot{y} = -\sqrt{1 - e^{-\frac{2}{y}}} \tag{5.15}$$

Then, from

$$\frac{dx}{dy} = \frac{\dot{x}}{\dot{y}} = -\frac{e^{-\frac{1}{y}}}{\sqrt{1-e^{-\frac{2}{y}}}}, \tag{5.16}$$

by using Taylor expansion, we have its image in Fig. 4 (The vertical axis is x and the horizontal axis is y).

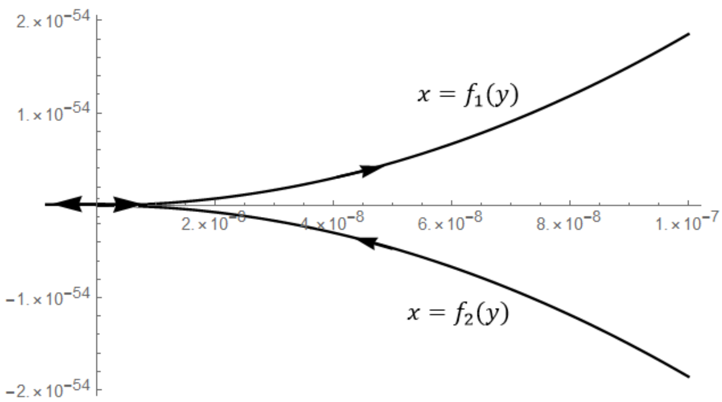

Fig. 4 – Branching of geodesic (one side reversible).

Remark 5.1 *The Finsler metrics Eq.* (5.1), *Eq.* (5.5) *of Examples* (A), (B) *are analytic functions on* $TM \setminus \{0\}$, *however, Eq.* (5.10) *of* (C) *is not so. If there is a point which is like* (3) *of Theorem* 4.1, *the Finsler metric* F *is not analytic.*

Proof. We assume F is analytic and the point p is (3)-type of Theorem 4.1. Since F is analytic, all components H_j^i are also analytic.
We consider the following functions $f^i(t)$ on a geodesic $c(t)$

$$f^i(t) := H_j^i\big(c(t), \dot{c}(t)\big)\dot{c}^j(t) - \phi(t)\dot{c}^i(t), \tag{5.17}$$

where $\phi(t)$ is an eigenvalue of the matrix $H_j^i\big(c(t), \dot{c}(t)\big)$ and also analytic. These functions $f^i(t)$ are all analytic.

We set $p = c(0)$ at $t = 0$. $f^i(t)$ is an Taylor expandable function on a certain interval $(-\delta, \delta)$ $(\delta > 0)$ because of analytic. Then

$$f^i(t) = f^i(0) + \frac{df^i}{dt}(0)t + \frac{1}{2}\frac{d^2f^i}{dt^2}(0)t^2 + \cdots + \frac{1}{n!}\frac{d^nf^i}{dt^n}(0)t^n + \cdots \tag{5.18}$$

is satisfied. The geodesic $c(t)$ divides at point p, so $c(t)$ is reversible on $(-\delta, 0]$ and not reversible on $(0, \delta)$. Therefore, on $(-\delta, 0]$, $H_j^i\big(c(t), \dot{c}(t)\big)\dot{c}^j(t) = \phi(t)\dot{c}^i(t)$, namely, $f^i(t) \equiv 0$ is satisfied. We have

$$f^i(0) = \frac{df^i}{dt}(0) = \frac{d^2f^i}{dt^2}(0) = \cdots = \frac{d^nf^i}{dt^n}(0) = \cdots = 0\,. \tag{5.19}$$

On the other hand, on $(0, \delta)$, $H_j^i\big(c(t), \dot{c}(t)\big)\dot{c}^j(t) \neq \phi(t)\dot{c}^i(t)$ is satisfied. Therefore $f^i(t) \neq 0$ is hold. However, from Eq. (5.19), $f^i(t) = 0$ $(\forall t \in (0, \delta))$ is induced, but this is contradiction. Therefore F is not analytic. *Q.E.D.*

Theorem 5.1 *Let M be an analytic manifold. If Finsler metric F is analytic, then each geodesic is only* (1)-*type of Theorem* 4.1, *namely, not reversible or only* (2)-*type, namely, reversible.*

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NOTE ASUPRA REVERSIBILITĂȚII ȘI RAMIFICĂRII GEODEZICELOR ÎN SPAȚII FINSLER

(Rezumat)

În acest articol sunt studiate condiții de existență pentru geodezicele reversibile și curbele *lpd*-simetrice, folosind noțiunea de transport paralel liniar. Sunt obținute, în mod special, condiții de existență locală ale acestora prin investigarea condițiilor lor de integrabilitate. În plus, este studiată și ramificarea geodezicelor.