BULETINUL INSTITUTULUI POLITEHNIC DIN IAŞI
Publicat de
Universitatea Tehnică „Gheorghe Asachi” din Iaşi
Volumul 65 (69), Numărul 1, 2019
Secţia
MATEMATICĂ. MECANICĂ TEORETICĂ. FIZICĂ

FOUNDATIONS OF SKYRME ANSATZ

BY

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Received: February 26, 2019
Accepted for publication: March 28, 2019

Abstract. Skyrme ansatz, that brought the 2×2 matrices into the nuclear realm merely by the idea of isospin, turns out to fill in naturally for one of the most important notions left behind by Newton in the definition of central forces. Newton definition uses the measurement idea, and takes care of the comparison of forces acting on different directions in space. It leaves uncovered what happens along the same line in space, an issue that came to fall under the third principle of dynamics. The Skyrme ansatz, as a natural completion of the Newtonian philosophy is actually the expression of the measurement along the same direction.

Keywords: Skyrme theory; Skyrme ansatz; Newtonian theory.

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1. The Importance of a Theory of Nuclear Matter

One can safely say that the nuclear matter is the one and only place where the matter meets the space in the most direct way that can ever be described in the science based on perception of matter by human means. Likewise, here is the place of the highest degree of uncontrollability over the forces with which we are accustomed in theoretical physics. However, here the prevailing treatment of uncontrollability today is done, as a rule, by means which Newton left behind in the treatment of force. We mean first the collisions, which assure the analogy with the earthly practice, allowing us to introduce the forces in the heavenly matters. This is perhaps the deep reason why the human spirit found necessary to reinvent the Hertz's particle as 'parton' (Feynman, 1969) within the energetic formalism of particle physics. The parton model works at very high energies, where the partons can be considered almost free with respect to each other – of course, with a specific understanding of freedom through a Feynman diagram – in the structure of a nucleon (see also (Drell, 1970)).

Here, in discussing the structure of nuclear components, the tough problem of deconfinement is heatedly debated even today, a half-century after it was initiated (Drell, 1977) for a clear early presentation of the idea of confinement and deconfinement). The first point at issue is that the quarks – the hypothetical fundamental constituents of the nuclear matter – cannot be free: they are confined inside nuclear matter with forces of a magnitude increasing with the distance from the center of nucleus. They can be 'deconfined' in some extreme conditions involving our ideas of ideal gas and hot plasmas (Satz, 1986), existing only in hypothetical stars or hypothetical cosmogonic events. The same goes obviously for partons, so that it is not at all a rare occurrence, in theoretical physics today, the question if quarks are partons or vice versa.

Inasmuch as the elucidation of the problem of deconfinement is thought as a thermodynamical problem, one can say that it will never get a definite answer: the thermodynamics, per se, does not qualify for a principle (Mazilu and Agop, 2012), regardless of the fact that deconfinement refers to quarks, partons, or even to any other conceivable components of the nucleus. This should have been even intuitively obvious: a solid body can go into liquid, and further into gas, by raising the temperature. We can understand this, and explain it, by inventing some forces of the van der Waals type, for these work here exactly as they were supposed to work in the first place (van der Waals, 1899): *for extended material points*. However, we are not able to understand completely how, continuing to raise the temperature, the gas can go into a plasma. This would require, first and foremost, Newtonian forces, and the thermodynamics cannot properly account for them. They were eliminated from the thermodynamics of ideal gases simply because the laws of these were extracted based on experiments in which the natural state is an externally confined one. One may say that the confinement issue has occurred in the nuclear realm strictly because here the laws of thermodynamics do not work anymore: for thermodynamics the confinement is implicit in the very definition of the temperature, and it is this one which needs primarily a careful revision.

But we might digress here for a long while, because the main issue is not a thermodynamical one, but purely philosophical if we may say so: the partons and quarks are never the same! The partons are those that may be taken as equivalent to the molecules of a gas, while the quarks are formal things accounting for the statistics of the 'inter-partonic forces' and described by a gauge theory of the kind we presented above. Let us elaborate on this issue.

The idea here is that today's partons are the direct relatives of old Hertz particles. They are always related by forces – more precisely, central forces – and we can say that the modern theoretical physics just found a way to look at them as if they were free. However, if we are to look at them as belonging to nuclear matter, then the harmonic maps are involved, and therefore the relative coordinates of these particles within nuclear space are harmonic coordinates, equivalent with forces. One can describe the confinement as a fundamental aspect of matter, but only in terms of partons (or Hertz's particles), in the following way.

The relative coordinates of the particles inside nucleus are equivalent to forces, elastic at first. This statement can be even reinforced by the observation that such forces should be somehow tied up with the metric of the ambient space, and they are indeed: they can be presented as Killing vectors of the Euclidean metric. But there is more to it: at a certain scale, these forces need to be described as stresses, leading to the idea of gauge as it was presented in (Mazilu and Agop, 2012). Which brings us to the suggestion of a description of matter in the manner from (Mazilu and Agop, 2012), taking again the gauge theory of light as a guide.

Indeed, the constitutive relations from (Mazilu and Agop, 2012) are only particular cases of the natural relations between the roots of two cubic equations. These are homographies, and the group relating two cubics is the Baker's group (Burnside and Panton, 1960; Mazilu and Agop, 2012). The constitutive relations as written in terms of eigenvalues, for instance (Mazilu and Agop, 2012), are only special case of homographies, whereby the transformation parameters are small. This is the circumstance that allowed the description of ether in vacuum as a special kind of matter, in a Witten-type ansatz. As we have seen, it turns out to be a fair description of light, from an electromagnetic point of view, as expected.

Now, the confinement of particles (partons) should be expected as a special property of matter, but the quadratic approximation of the constitutive law does not allow it: there is not a vertical or horizontal asymptote of that law. However, the homography has such a property, by its very definition, so that we can accept such a function as a constitutive law inside the matter, and so much more in the nucleus. From this point of view, the Manton geometrization allows for a general expression of the Skyrme principle, with algebraically homogeneous functional. This will be now briefly described.

The quadric is the starting point of Dan Barbilian in the construction of the Riemann spaces associated with families of one-parameter cubics, as Cayley-Klein spaces (Barbilian, 1938; Mazilu and Agop, 2012). The geometrical procedure used by Barbilian will be now discussed in broad lines, but with a special physical interpretation coping with the modern idea of gluons. First, we need to notice that Barbilian begins with the idea that the starting quadratic form is the Hessian of a cubic with real roots, therefore it has complex roots. Then, by performing the linear transformation of homogeneous coordinates

$$\frac{X_0'}{uX_0+X_1}=\frac{X_1'}{u^2X_0-X_2}=\frac{X_2'}{\left(u^2+\frac{v^2}{2}\right)X_0-X_2}$$
$$=\frac{X_3'}{u\left(u^2+\frac{v^2}{2}\right)X_0-\left(u^2+\frac{v^2}{2}\right)X_1-uX_2+X_3}$$

we can reduce the quadric to a ‘canonical’ form

$$X_0'X_3'-X_1'X_2'=0$$

showing explicitly that we have to deal with a one-sheeted real hyperboloid. Using now the Barbilian form of the roots of a cubic, one can show that the set of all cubics having real roots is isomorphic with an ensemble of the oscillators having the same frequency.

First, a 2×2 matrix representation of a cubic is obvious even from the Barbilian form of the solutions of a cubic with real roots. Indeed, the same transformation is applied to the ‘fixed point’, *i.e.* the triplet $(1, \varepsilon, \varepsilon^2)$, to get the ‘current point’, *i.e.* the triplet (x_1, x_2, x_3). Therefore the cubic having these roots can be faithfully represented by that transformation, which can be represented as a point in space, representing either a cubic equation or a 2×2 matrix, or both at once, as the case may occur:

$$\frac{X_0'}{h^*k}=\frac{X_1'}{h}=\frac{X_2'}{k}=\frac{X_3'}{1} \tag{1}$$

Even though this is not a real matrix, it can be framed in the geometry discussed in (Mazilu and Agop, 2012). In fact, it can be shown that this representation is in close connection with the Iwasawa decomposition of a 2×2 real matrix. Thus, the Cayley-Klein (absolute) metric of this representation is the Barbilian metric given by

$$(ds)^2 = \left(\frac{dk}{k} - \frac{dh + dh^*}{h - h^*}\right)^2 - 4\frac{dhdh^*}{(h - h^*)^2} \tag{2}$$

where h and h^* are the roots of the Hessian and k is an arbitrary complex factor of unit modulus. This metric is also characteristic to a genuine characterization of an ensemble of oscillators, starting from their classical description by the solutions of a second order differential equation.

The second order differential equation characterizing a classical damped harmonic oscillator, can be written as

$$M\ddot{q} + 2R\dot{q} + Kq = 0 \tag{3}$$

with obvious notation for first and second derivatives of the relevant coordinate q. The solutions of Eq. (3) form a two-dimensional manifold depending on *three arbitrary parameters*. They can be written in the form

$$q(t) = e^{-\lambda t}(he^{i(\omega t+\phi)} + h^* e^{-i(\omega t+\phi)}); \quad M^2\omega^2 \equiv MK - R^2, \quad M\lambda \equiv R \tag{4}$$

Therefore, the solutions represent an ensemble of oscillators of the same frequency, in which the element is identified by three parameters h, h* and $e^{i\phi}$. Having the experience of the blackbody radiation, one might say that the frequency is somehow statistically related to this ensemble of oscillators, the same way the temperature is related to the kinetic energy, assuming of course that it is possible to find such a statistic. This statement can even be made more precise from algebraical point of view.

Indeed, the ratio of any two linearly independent solutions of the differential Eq. (3), τ say, is a solution of the following differential equation

$$\{\tau, t\} = 2\omega^2 \tag{5}$$

where the curly brackets denote the so-called Schwartz derivative of τ with respect to time, defined by (Mihăileanu, 1972)

$$\{\tau, t\} \equiv \frac{d}{dt}\left(\frac{\ddot{\tau}}{\dot{\tau}}\right) - \frac{1}{2}\left(\frac{\ddot{\tau}}{\dot{\tau}}\right)^2 \tag{6}$$

This differential expression, and therefore the left-hand side of the Eq. (5) along with it, is invariant with respect to the homographic transformation of the function, *i.e*. the ratio of two fundamental solutions of the Eq. (3)

$$\tau \leftrightarrow \tau' = \frac{a\tau + b}{c\tau + d} \tag{7}$$

with a, b, c, d four real parameters. The set of all transformations (7) corresponding to all the possible values of these parameters is obviously the group SL(2, R).

Thus the ensemble of all oscillators of the same frequency is in a one-to-one correspondence with the transformations of SL(2, R). This allows us to construct a 'personal' parameter τ, so to speak, for each oscillator of the ensemble of possible solutions of the classical equation of the harmonic oscillator, guided by the form of the general solution of Eq. (5). This solution can indeed be written as

$$\tau' = u + v\tan(\omega t + \phi) \tag{8}$$

where u, v and ϕ are constants, characterizing a given oscillator from the ensemble as before. Identifying the phase from (8) with that from (4), we can write the 'personal' parameter of an oscillator in the form

$$\tau' = \frac{h + h^{*}\tau}{1+\tau}; \quad h \equiv u + iv; \quad h^{*} = u - iv; \quad \tau \equiv e^{2i(\omega t+\phi)} \tag{9}$$

This equation reveals the Barbilian form of the roots of a cubic equation, so our results can be summarized as follows:

1) each oscillator represents a family of cubic equations depending on time

2) the initial conditions of the oscillator – the amplitudes – are given by the roots of the Hessian of the cubic family.

In physical terms this means that the ensemble of oscillators represents a family of matrices giving a measured field. The physical quantities accessible to measurement are the normal and shear components of the field. The time of physical evolution of the field is given by the phase angle of orientation of the octahedral vector in the local octahedral plane.

Thus, the ensemble of initial conditions of the oscillators corresponding to the same frequency can be organized as a geometry of the hyperbolic plane in the representation of Poincaré (Mazilu and Agop, 2012). Therefore, these oscillators correspond to a situation where their initial conditions can be chosen from among the points of a hyperbolic plane. One such situation is that of the Kepler motion representing an atom. Therefore, these oscillators are related to the structure of the nucleus, and with the stresses inside nucleus. In other words, the general constitutive law enforces naturally by the representation of stresses and strains represented as 3×3 matrices has a dynamical interpretation in terms of oscillators.

The metric (2) offers now a natural posibility of extension of the harmonic principle that we associated with a Kepler problem, by a functional which in fact represents the deformation of the nuclear matter in the general case. The extension amounts to expressing the energy of mapping by the functional

$$E(\mathbf{\Phi}) = \iiint \left\{ -4\frac{(\nabla h \cdot \nabla h^*)}{(h-h^*)^2} + \left(\frac{dk}{k} - \frac{\nabla h + \nabla h^*}{h-h^*} \right)^2 \right\} (d^3\vec{x}) \tag{10}$$

The variational principle would then express this simple fact: the most general deformation of the nuclear matter is replicated in the atomic structure by the variation of the eccentricity of the electronic orbits. Likewise, inside nucleus, the variation is replicated by an ensemble of harmonic oscillators of the same frequency. The second term from this functional (Mazilu and Agop, 2012) is analogous to the baryonic term from the theory of Skyrme, with the only difference that now the whole functional is homogeneous. In the language of skyrmion technology however (Atiyah and Sutcliffe, 2001), it still represents hyperbolic skyrmions, because the geometrical character of the problem is dictated by the metric from Eq. (2).

In this form, the theory illustrates the strong ties of the model of nuclear matter with the ideas of confinement of matter in general. In order to show this, we will present now a solution to the variational principle related to the energy functional from Eq. (10). In order to better understand intuitively that solution, let us start with a ‘strange’ classical dynamics, the one related to the forces involved in the problem of confinement of the classical ideal gas (Mazilu and Agop, 2012), of magnitude inversely proportional with the distance between molecules. This force is also involved in a Newtonian description of the inertia (Sciama, 1969) which, again should appear as quite natural in view of the common background of the general relativity and Skyrme theory (Mazilu and Agop, 2012).

Assuming a Newtonian dynamics for this force, the equations of motion can be written in the form

$$\ddot{\vec{r}} + \frac{\mu^2}{r^2}\vec{r} = \vec{0}$$

This is a plane motion, because the force is central. In polar coordinates of the plane of motion, the equations of motion splits into the system of two differential equations

$$\ddot{r} - r\dot{\theta}^2 + \frac{\mu}{r} = 0, \qquad r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \tag{11}$$

The second of these gives the area constant:

$$r^2\dot{\theta} = \dot{a}$$

The first Eq. (11) can then be integrated as follows (Govinder *et al*., 1993): first change the time variable into

$$\tau(t) = \int \frac{dt}{r^2} \quad \therefore \quad \theta'(\tau) = \dot{a} \tag{12}$$

where the prime represents differentiation with respect to τ. Now, if we change the dependent variable into $\xi(\tau) = 1/r$, the first Eq. (11) becomes

$$\xi'' + (\dot{a}^2 - \mu^2)\xi = 0; \quad \xi(\tau) \equiv \frac{1}{r} \tag{13}$$

This is the equation of a harmonic oscillator, with the frequency dictated by the rate of area, leading to either trigonometric or hyperbolic functions. We consider only the first case, when the solutions are of the form

$$\xi(\tau) = A\sin(\Omega\tau) + B\cos(\Omega\tau), \qquad \Omega^2 \equiv \dot{a}^2 - \mu^2 > 0 \tag{14}$$

Now we can find the relation between the Newtonian time and the new time τ. Indeed taking Eq. (14) into Eq. (12) gives

$$t \equiv \int \frac{d\tau}{\xi^2(\tau)} = \int \frac{d\tau}{\left[A\sin(\Omega\tau) + B\cos(\Omega\tau)\right]^2} \tag{15}$$

and this integral leads to (Gradshteyn and Ryzhik, 1994)

$$\Omega\tau = \phi_0 + \tan^{-1}\left[(A^2 + B^2)\Omega t\right] \tag{16}$$

where ϕ_0 is a constant of integration. Interestingly enough, Eq. (14) represents a particular Kepler motion, corresponding to null gravitational constant, or null mass, or even null charge as it were. It cannot be therefore interpreted in terms of the motion of a material point around another attracting material point. However, it can be interpreted in terms of an abstract kinematics, suggested by the analogy between Eq. (16) and the solution (8) of the Schwartzian equation, which can be interpreted as the eigenvalue of a stress matrix.

Consider indeed the kinematics generated by differential forms from (Mazilu and Agop, 2012). In terms of these the metric (2) assumes the Lorentzian form. Along the geodesics of this metric the rates represented by the above-mentioned differential forms are constant, so that we can find those geodesics from some differential equations involving a parameter linear in the arclength from Eq. (2). These are

$$\begin{aligned} & d\phi + \frac{du}{v} = a \cdot dt, \\ & \cos\phi \frac{du}{v} + \sin\phi \frac{dv}{v} = b \cdot dt, \quad -\sin\phi \frac{du}{v} + \cos\phi \frac{dv}{v} = c \cdot dt \end{aligned} \tag{17}$$

with a, b and c some constants. The last two of these equations give

$$\frac{\dot{u}}{v} = b\cos\phi - c\sin\phi, \quad \frac{\dot{v}}{v} = b\sin\phi + c\cos\phi \tag{18}$$

and then from the first of (17) we have

$$\dot{\phi} = a - b\cos\phi + c\sin\phi \tag{19}$$

an equation which can be integrated right away. We prefer to perform this integration by putting the right-hand side of (19) in the form of a perfect square, in order to show that this corresponds to the Eq. (15) above. Indeed, if we take $2\Omega\tau \equiv \phi$, we can write the integral in the form (16) for a, b, c given by

$$a \equiv \frac{A^2 + B^2}{2}, b \equiv \frac{A^2 - B^2}{2}, c \equiv AB \tag{20}$$

This gives an interpretation to the classical 'time' variable τ, provided we know something about this abstract kinematics. Insofar as the parameters u and v are concerned, using Eq. (18) we have

$$\frac{v}{v_0} = a - b\cos\phi + c\sin\phi \tag{21}$$

where v_0 is another integration constant. Therefore, v represents the inverse square of the position vector of the motion previously described. On the other hand, for the parameter u we find the following solution

$$v_0(u - u_0) = b\sin\phi + c\cos\phi \tag{22}$$

where u_0 is another constant of integration.

One can find now a particular solution of the variational principle applied to the energy functional (10) along the geodesics given by Eqs. (21) and (22), if we assume that their parameter (and therefore the phase ϕ) is a solution of the Laplace equation. This can be proved, the easiest way, by continuing to work in real parameters u, v and ϕ as before. The Euler-Lagrange equations associated with the variational principle applied to (10) are:

$$\nabla \cdot \left(\nabla\phi + \frac{\nabla u}{v} \right) = 0; \quad \nabla^2\phi - \nabla\phi \cdot \frac{\nabla v}{v} = 0; \quad \nabla^2(\ln v) - \nabla\phi \cdot \frac{\nabla u}{v} = 0 \tag{23}$$

On the other hand, the geodesics of the metric (2) are solutions of the system of differential equations:

$$\left(\frac{u'}{v} \right)' + \phi' \cdot \frac{v'}{v} = 0; \quad \phi'' - \phi' \cdot \frac{v'}{v} = 0; \quad (\ln v)'' - \phi' \cdot \frac{u'}{v} = 0 \tag{24}$$

where the prime means differentiation with respect to the parameter of geodesics. Now, if the functions u, v and ϕ depend on position only through the parameter of geodesics, the Eqs. (23) can be written in the form

$$\left(\phi' + \frac{u'}{v} \right)' (\nabla t)^2 + \left(\phi' + \frac{u'}{v} \right) \nabla^2 t = 0; \quad \left(\phi'' - \phi' \cdot \frac{v'}{v} \right) (\nabla t)^2 + \phi' \nabla^2 t = 0$$

$$\left((\ln v)'' - \phi' \cdot \frac{u'}{v} \right) (\nabla t)^2 + (\ln v)' \nabla^2 t = 0$$

The first terms of these equations are zero as a consequence of the equations of geodesics. So, we have still another form of the harmonic principle: in fairly general conditions, the harmonic mapping corresponding to energy (10) is given by the geodesics of the metric, provided their parameter is a regular harmonic function. This of course makes the phase ϕ the arctangent of such a function, in view of the Eq. (19) above.

Therefore, at least in this particular instance, we have to focus on the geodesics of the metric (2). The Killing vectors of the metric represent conservation laws, and thus the parameters a, b, c from Eqs. (21) and (22) represent are expression of these conservation laws. The Eqs. (21) and (22) themselves represent two Kepler motions, and by this we are certainly in position to know exactly the field of application of the above kinematics: it is the theory of space stresses, involved in the Kepler problem. These stresses are induced in the core of the solar system for instance, or in the nucleus, and they represent a material point – in the sense of Hertz – whose particles, acted upon by the inverses of the space elastic forces, behave like an ideal gas. The density of this material point varies inversely proportional with the square of distance from the origin of the reference frame, which is actually its origin. We have to insist upon these two aspects of the problem of action at distance, for they are instrumental in understanding how the nucleus works.

It is well-known that one of the first theories of nuclear matter was that of a gas model (Fermi, 1950). It was not so successful, but certainly touched a fundamental aspect of the problem of structure of nuclear matter, which turns out to be actually the fundamental problem regarding the structure of the matter in general. In view of the discussion above, we may assume the following scenario: the particles of nucleus are decided ‘in pairs’ by the inversion transformation between external Newtonian forces and the forces, internal to the nucleus, of confinement. These last ones are represented as harmonic oscillators – gluons as it were – and described by a general dynamics. The kinematics of a pair is represented as two Kepler motions given by Eqs. (21) and (22) above. Here, of course, we assume that the pair is ‘accidentally decided’ by inversion, but once decided it is described by an actual state of stress whose kinematics can be classically described. This state of stress can be physically characterized by a statistics of the kind specifically connected with the light (Mazilu and Agop, 2012).

With the Eqs. (21) and (22) above, we certainly can extend to matter, particularly to the nucleus, the Mac Cullagh’s view about the structure of light (Mac Cullagh, 1831). Indeed, the two equations represent two harmonic oscillators, as well as two Kepler motions. In classical terms, one can say that two particles (partons) inside nucleus have independent motions of a Kepler kind. Even though classically described, such an image should be statistically accessible to measurement through some kind of stresses or strains, and reflected in the eccentricity of electronic orbits.

This image of the structure of the nucleus can be made even more precise, from the very classical point of view we advocate in this work. Indeed, inasmuch as we are thinking in the framework of the Newtonian natural philosophy, the above view on the Hertz’s particles – the partons of nuclear matter – recovers, and solves we should say, in a ‘quantum-mechanical’ way, one of the most important issues of that philosophy, left behind by Newton in his theory of forces.

2. Newtonian Foundation of Skyrme Ansatz

According to (Mazilu and Agop, 2012), where it has been shown that the classical description of the Kepler motion leaves room for an elegant description of the nucleus through the geometry of the hyperbolic plane, in terms of eccentricity and orientation of the Kepler orbit. More to the point, if we express the complex number h, representing the asymptotic direction of the orbit, as a function of the eccentricity e of the orbit and its orientation, α say, then the absolute metric obtained as a result of the constraint which expresses that the Kepler orbit is a closed conic section, is given by

$$(ds)^2 = (d\psi)^2 + \sinh^2\psi(d\alpha)^2; \quad e \equiv \tanh\psi \tag{25}$$

The angle α gives the orientation of the orbit in its plane with respect to an arbitrary direction. If ψ is a regular harmonic function in space, then the complex number h gives a mapping from space to the hyperbolic plane. Therefore, if the nucleus is made of Hertz particles and we identify the harmonic coordinates of these particles with the forces between them (Mazilu and Agop, 2012), then the eccentricity of this orbit is a physical expression of the statistics of intranuclear forces between constituent particles of nucleus.

The metric (25) is also the metric of a section of hyperbolic space, used by Atiyah and Sutcliffe, for instance, in the construction of hyperbolic skyrmions. The form of this metric is, in general

$$(ds)^2 = (d\psi)^2 + \sinh^2\psi(d\theta^2 + \sin^2\theta d\phi^2) \tag{26}$$

where θ and ϕ are usual spherical polar angles. Obviously, (25) can be obtained from (26) if we agree that α represents the geodesic arc on the unit sphere. But, as it was shown in (Mazilu and Agop, 2012), Eq. (26) is also the absolute metric of the space of relativistic velocities (the Fock metric) and probably there are still many meanings of it, of which one is of a particular cosmological interest.

It is indeed, worth mentioning the fact that, if it is to be applied to the interior of atomic nucleus, the theory of gravitation in its Einsteinian form can only refer to a cosmology. More than this, this cosmology has to be spatially homogeneous, in the sense of existence of a transitive group describing it. Now, inasmuch as this homogeneity involves a three-parameter group, with the structure of Barbilian group, the homogeneous space should be of the Bianchi type VIII in the language developed by cosmologists (Bianchi, 2001; Taub, 1951). Consequently the metric from Eq. (2) above, must have some obvious connections with the metric of some spatially homogeneous cosmologies. One such cosmology aiming explicitly to the introduction of the material points in the form of galaxies, is the Gödel's cosmology, taking into consideration the nonzero density of matter by a cosmological term in Einstein's equations (Gödel, 1949; Gödel, 1952). The metric of such a space time is given by

$$(dx_0 + e^{x_1}dx_2)^2 - (dx_1)^2 - \frac{e^{2x_1}}{2}(dx_2)^2 - (dx_3)^2 \tag{27}$$

with the usual relativistic notation taking x_0 as the time coordinate and $x_{1,2,3}$ as space coordinates. One can see that the section with constant x_3 of this space-time has a metric almost identical with the metric from Eq. (2). In fact it can be written in the form given in Eq. (2) by a transformation which offers 'physical meaning', so to speak, to time and plane coordinates:

$$x_0 = \sqrt{2}\phi; \quad x_1 = \ln\frac{\sqrt{2}}{v}; \quad x_2 = u \tag{28}$$

Here we used our usual notation, $h \equiv u + iv$, $k \equiv e^{i\phi}$. Rotating cosmologies carry the particular significance of an universe in motion, but with important proviso that the time is given by this motion, it is not a priori parameter. In this specific case, one can say that Gödel cosmology, when applied to 'nuclear universe', gives a dynamic explanation of this universe in terms of the classical motions of its particles. We like to point out that this 'dynamic explanation' can be done in terms of Kepler motions as geodesics of the Barbilian metric – Eqs. (21) and (22) above. In fact this might justify the point of view that the Barbilian space is a phase space for the Kepler motion in general. In the case of Kepler motion the velocity vector follows a circle while the moving body follows the conic describing the motion (Milnor, 1983). One can assume that, in general, the dynamics of the motion is described in a phase space and the trajectories are represented by two conics. If the Eq. (21) represents a conic:

$$\frac{1}{r} = a - b\cos\phi + c\sin\phi \tag{29}$$

then this radial motion has a speed given by

$$\dot{r} = b\sin\phi + c\cos\phi \tag{30}$$

in view of the time transformation from Eq. (12). Here the time derivative of the radial coordinate is taken with respect to the very time of the motion, as it should be. Eliminating the angle between Eqs. (29) and (30), one can even get a conservation law. This simple fact shows that, the radial component of the Kepler motion is actually of the same nature – at least from an algebraic point of view – with the free radial motion (Mazilu and Agop, 2012). But there is more to this interpretation, even outside the Hamiltonian theory: it touches the very essence of the classical Newtonian theory of forces.

The metric from Eq. (26) can be brought to the form (2), for both are metric of constant negative curvature. As shown above, one just needs for this a transformation offering physical meaning to coordinates. Now, the fact that metric (26) was used in describing the skyrmions with zero mass pions, incites us in constructing for it a fundamental skyrmion with distinguished significance in the very roots of Newtonian mechanics. For that skyrmion, the Eq. (20) – and therefore Eq. (2) – still maintains the meaning of absolute metric (Mazilu and Agop, 2012), but this time it is also related to a conservation law. Let us follow closely the logic of such a construction, starting even from the Newtonian theory of forces.

Fact is that Newton's definition for the central force, the kind of force needed in astronomical researches as well as in the theory of nuclear particles, is actually a definition based on the concept of measurement of forces. We stressed this idea quite a few times in the present work. It amounts basically to the fact that the ratio of magnitudes of forces acting on a planet in different directions from the plane of motion, but determining the very same Keplerian orbit, covered in the same time interval, can be recognized in the elements of that orbit (Mazilu and Agop, 2012). This is the essential point of the Corollary 3 of the Proposition VII from *Principia*, which contains the most general definition of the gravitational force, leading to the idea of its 'universality'.

There is, however, an essential point left uncovered by Newton, and this is the problem of mass. He insisted in the proportionality of the central force of heavens with the product of the masses involved in interaction, which is quite an arbitrary assumption. It is not wrong, by any means – the history proves it – but, we should say, incomplete. It was obvious even to Newton himself that in order to be logically correct, the hypothesis was quite insufficient as it stood, and needed to be amended with 'perturbation' terms in order to account for the fact that the Sun is not fixed, and we do not deal here with material points without space expanse (see for details (Popescu, 1988)). Nevertheless, used as Newton gave it first, the theory of forces led in a natural way to the equation of Poisson (Mazilu and Agop, 2012), which further led to the general relativity.

Newton's principle of measurement compares the forces *along two different lines* from the plane of motion of the planet, in a ratio depending only on the geometrical elements of the motion (Mazilu and Agop, 2012). In the long run, one might say, let us stress it once more, that his philosophy is simply the one saying that the ratios of the forces, acting upon a planet in any two directions in plane, are to be read in the planet's motion. The force must be always central, otherwise in the vector model of forces the orbit is no longer plane. In view of todays astronomical knowledge this seems to be quite an ideal speculative conclusion, for no motion in the universe seems to be plane. However, in the times of Newton the only thing to be taken into consideration was the Kepler's synthesis of planetary observations, which plainly sustained the conclusion of plane motion. This is why we take special precaution of talking of Keplerian orbits, whenever we have to discuss the subject of Newtonian forces: they are related exclusively to the Keplerian setup of the geometry of motion.

However, Newton's hypothesis about masses in the expression of the magnitude of forces, aims to describe, and introduce within the concept of central force, not what happens *along two different lines* of action in plane, but what happens *along the same line* of action. In view of the reciprocity of gravitational interaction, the action at distance is the same no matter of the point of view along its line of action, and it should therefore be characterized by the magnitude of the vector of force. Moreover, by the very same token, this magnitude should not depend on the direction of action along the same line. The product of masses in the expresion of force is an algebraical monomial satisfying this requirement, but so is the sum, or any kind of average for that matter, like for instance the reduced mass. Later on, even the equations of motion of the classical dynamics led to the conclusion that the masses are only convenient coefficients, and their choice is only justified by the simplicity of mathematical description (Poincaré, 1897).

As we see it, the point at issue is very probably the fact that the third principle of dynamics is no more actual *over the space*. If the forces do not act in the same point and, moreover if, physically speaking, this point does not remain unchanged by them, the forces cannot be compared. In other words, the point where forces act should be a Hertz particle. This is hardly the case with the matter in general, but even in case it would be such a case the central action still needs to be amended. Indeed, the quantity of matter *must be* essential. However, one cannot say that a planet perceives in the Sun the same amount of matter that the Sun itself perceives in the planet. In other words, the fact that the masses involved in interaction are different should be inherently contained in the definition of the very magnitude of forces.

Then this part of the definition of forces should be taken out of the classical vector formalism, which apparently does not allow for a proper accomodation of the concept of mass. And it can be taken indeed, by means of a quantum definition of the measurement, in the axiomatic manner in which the isospin is introduced in the theory of nuclear forces (Rosenfeld, 1948), or the spin is introduced in quantum mechanics (Schwartz, 1977). Namely, the situation just described is represented by the following hermitian 2×2 matrix depending explicitly on the direction in space – vector characteristic – but having two eigenvalues equal in magnitude, independent of that direction:

$$\mathbf{Q} \equiv \begin{pmatrix} \cos\theta & \sin\theta e^{i\phi} \\ \sin\theta e^{-i\phi} & -\cos\theta \end{pmatrix} \tag{31}$$

This matrix has, indeed, the eigenvalues ±1, or any two real numbers equal in magnitude and opposite in sign for that matter. Consequently it can represent the masses of two ideal particles along any direction in space, acting on each other with a central Newtonian force. Based on matrix (31) we can build an *ansatz*. First notice that any 2×2 matrix of the form

$$\mathbf{M} = \lambda\mathbf{E} + \mu\mathbf{Q} \tag{32}$$

where λ and μ are real, and **E** is the 2×2 identity matrix, has two different eigenvalues not depending on the direction. These are (λ ± μ). The ansatz is then exactly the *Skyrme ansatz*, inasmuch as Eq. (30) can be written in the form

$$\mathbf{M} = \exp(\psi \mathbf{Q}) \tag{33}$$

The matrix **M** then represents indeed a skyrmion, because it is referring to quantities that characterize the nucleus. First, the absolute metric (Mazilu and Agop, 2012) of the matrices (32) or (33) is just the metric from Eq. (26), where we only need to take

$$\frac{\mu}{\lambda} = \tanh \psi \tag{34}$$

What this representation tells us is that the measurement provides the masses ($\lambda \pm \mu$) of two particles in interaction, which they perceive into each other through this interaction, and which are independent of the direction (θ, ϕ) of the straight line joining them. The formula (34) is then justified by the inequality of the masses of the two partners involved in interaction. But this does not mean that the masses themselves are independent of the environment of the particles inside nucleus. Indeed, in order to represent the nuclear matter, ψ needs to be a solution of the Laplace equation over the space occupied by the nucleus. This certainly brings analytical dependence of the ratio of masses on the whole environment of the line joining the two particles. Moreover, by looking at the model presented in (Mazilu and Agop, 2012), ψ must be a force.

It is hard to see a situation illustrating this idea in the macroworld. However let us accept for now that there is such a situation, but we are not yet able to see it. For, in the microworld it is, so to speak, a common situation. The history of physics plainly illustrates it, which is why we sustain that the Skyrme ansatz is actually only a natural finish of the classical natural philosophy of forces. Indeed, take the case of proton and electron. In the Rutherford atom, described as a classical Kepler motion, they have significantly different masses. In this case, we can certainly say that the value of ψ in Eq. (34) is very small, the forces are elastic, and so the metric in Eq. (26) is very nearly Euclidean. This is why the atom could be described as a Kepler motion or as a harmonic oscillator in the first place. This description is legitimate even from the point of view of the isospin proper. Indeed, it naturally contains the limit case of quantities equal in magnitude and opposite in sign, that the electron and the proton perceive into one another in special occasions. This is, of course, the case of their charges which, as well-known, are equal in magnitude and of the opposed signs. The pairs of equal mass, for instance the proton and antiproton, or the electron and positron, are here represented by the limit of very large ψ, *i.e.* of the very small difference of their masses.

The theory of nuclear forces asks necessarily for a kind of algebraic theory of masses, whereby the idea of direction intimately enters the theory (for instance in the form of mixing angles). And the above approach shows plainly how the notion of direction should work, and where the ratio of masses intervenes. However this idea of richness of the notion of direction, and of its direct relation with the ratio of masses is not new, by any means. With specific reference to the quantum theory of fundamental Newtonian structures and their relation with cosmology and cosmogony, the idea was elaborated to a large extent by Eddington (Eddington, 1936; Eddington, 1953). Our only addition here is perhaps that the quantum theory has a direct descent from the classical Newtonian theory of forces, which it just completes and finishes naturally. The Skyrme theory seems to be indeed, the only right addition to that classical theory, in the sense of completing it in its very own terms.

3. Conclusions

The main conclusions of the present paper are the following:

i) Arguments for the importance of a theory of nuclear matter are discussed.

ii) A mathematical model according with Barbilian's differential geometry is developed.

iii) By employing this model, a Newtonian foundation of Skyrme ansatz is established.

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FUNDAMENTELE SKYRMIONILOR DE TIP ANSATZ

(Rezumat)

Se arată că teoria lui Skyrme este o alternativă viabilă la Principiile matematice ale filosofiei naturii a lui Newton. Într-un asemenea context, direcția şi nu distanța devine fundamentală în orice proces de măsură, independent de rezoluția de scală.